Corporate Debt Standardization and The Rise of Electronic Bond Trading

[PRELIMINARY DRAFT]

Artur Carvalho

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Abstract

I study the impact of standardization on secondary corporate bond markets as the industry adopts electronic trading systems. I show that covenants can reduce debt rollover costs by mitigating agency problems. However, when trading in the more liquid electronic markets is restricted to standardized securities, firms must weigh the benefits of offering credit protection against e-trading’s lower transaction costs. I investigate firms’ choices of leverage and debt type when creditors are not fully informed about their risk exposures nor their hedging policies. In such cases, riskier firms can have an incentive to misrepresent their types to benefit shareholders, which raises debt rollover costs for safer firms. Safer firms react by adjusting their leverage, either to signal their credit-worthiness and force separation, or to reflect the less favorable funding conditions in a pooling equilibrium. Alternatively, safe companies can signal their type by issuing bonds with debt protective covenants, leading to a separating equilibrium with a hybrid market structure, where safe bonds trade over-the-counter. I show that this is the case when the liquidity differential between over-the-counter and electronic markets is sufficiently low and risky types optimally choose not to hedge their exposure to their idiosyncratic risk in equilibrium.
I. Introduction

Despite a slow start, the $8 trillion US corporate bond market has seen a steady increase in electronic trading (e-trading) in recent years. This move has been hailed as a potential solution to the growing concerns about liquidity deterioration voiced by several institutional investors since 2013. In order to be successful, e-trading will require the standardization of newly issued bonds to aggregate liquidity in a few securities. Such move will limit the use of covenant clauses designed to prevent firms from exploiting their private information in pursuit of financial strategies that are detrimental to debt holders. This paper studies the trade-off between external market liquidity and the informational costs associated with the structural change in corporate bond markets.

I modify He and Xiong (2012) model to study the choice of debt standardization by firms as the industry moves towards electronic trading. In the model, illiquidity is proxied by portfolio liquidation costs. Covenant-free bonds can be transacted in the more liquid electronic platforms, whereas non-standardized debt can only be traded in over-the-counter (OTC) secondary markets. I show that covenants are important for mitigating agency problems and that standardization may decrease investors’ ability to more easily distinguish the credit quality of issuers.

I endogenize firms’ decision to issue bonds with debt protective covenants by introducing idiosyncratic, unhedgeable shocks that affect a subset of the firms. Absent any asymmetry of information, all firms issue standardized debt because the higher liquidity of electronic markets positively affects the valuation of their debt. However, when firms’ exposure to the unhedgeable shocks are not observable by bond investors, a conflict between bond holders and equity investors can arise. I show that some riskier firms may choose to deceive investors by issuing debt that is ex-ante indistinguishable from higher-quality debt. By doing so, these firms increase the rate of return to equity investors at the expense of debt holders. In response, safer firms adjust their leverage to either discourage riskier firms’ attempt to misrepresent their creditworthiness, or to accommodate a pooling equilibrium. Alternatively, safe firms might opt to issue bonds with a debt protective covenant to credibly signal their credit quality. The informational costs posed by debt standardization can thus offset the liquidity gains offered by the new trading technologies, leading to a smaller base of potential clients and reduced revenues for the new electronic markets.

The implications of the findings are two-fold. First, certain types of high-quality debt might still be traded in OTC markets, leading to differences in the composition of debt across secondary markets. Second, bond investors in electronic exchanges might have to trade liquidity for creditor protection.

A. Electronic Trading Venues and Bond Standardization

Since the 1940s, corporate bonds have been traded primarily over-the-counter (OTC). OTC markets are opaque and decentralized markets where dealers act as counterparties for both buyers and sellers. By setting the quotes at which they are willing to fulfill an order, they assume the trade execution risk and profit from the bid-ask spread. Liquidity in secondary corporate bond markets thus heavily depends on dealers’ ability to warehouse large inventories and their willingness to absorb customer order imbalances into their own balance sheets.

The low interest rate environment that has prevailed in the post-Crisis has led to a surge in corporate debt issuance, as companies have rushed to take advantage of favorable borrowing conditions. The principal amount outstanding in corporate bond markets grew from $5.5 bi in 2008 to $9 bi in a decade.¹ Dealers have profited along by underwriting these new issues. Despite the burst in corporate debt, however, dealers' inventories shrunk from $250 bi right before the Crisis to under

¹Source: Finra, available at https://www.sifma.org/resources/research/fixed-income-chart/
This deleveraging, along with the decline in block trading and anecdotal evidence of increased price impact for larger transactions in the secondary market, suggests a decrease in intermediaries’ risk appetite that has compromised their ability to provide liquidity.

Market participants have blamed the deteriorating liquidity conditions on post-crisis financial regulation. Designed to curb banks’ risk-taking and make the industry safer, these regulatory changes seem to have constrained dealer’s market making activity. The Basel III Accord and regulations under the Dodd-Frank Act of 2010 have imposed greater capital and liquidity requirements for banks, increasing their cost of capital and hampering their ability to maintain large corporate bonds inventories. In addition, the Volker Rule, which came into effect in April 2014 and prevents banks from engaging in “risky” proprietary trading, has led to the shut down of several proprietary trading desks in Wall Street. While standard measures of liquidity based on execution costs for completed trades, such as the bid-ask spread, appear healthy, search costs have risen. Consistent with Duffie (2012) assessment that the new bank rules would hurt dealers’ market making capacity, Bessembinder, Jacobsen, Maxwell, and Venkataraman (2018) find that bank-affiliated dealers have become less willing to commit capital and to accommodate block trades in recent years.

As a means of improving liquidity, some banks and large institutional investors have pushed for the modernization of secondary markets’ structure, with the adoption of new technologies aimed at cutting down costs and improving the efficiency of bond trading operations. The most disruptive and controversial change has been the shift towards electronic, equity-style trading on exchanges. Proponents have argued that electronic trading ameliorates trading conditions by reducing secondary markets’ dependency on intermediary capital. E-trading systems facilitate the direct matching of buyers and sellers because they improve the relaying and processing of information and allow customers to directly access several markets at once. In addition to partially replacing and improving upon basic broker services, these systems can help increase market transparency and lower entrance costs. More competition and improved market access then facilitate price-discovery, help restrict margins and reduce search costs by making it easier to find suitable counterparties to a trade.

E-trading has been gaining ground for the past five years. According to a recent report by

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4Corporate bonds were actively traded in the NYSE in the beginning of the 20th century. The exchange offered a high degree of pre- and post-trade transparency, since the book of available orders and recent trades were visible to all brokers. Trade abruptly migrated to OTC markets in the late 1940s. Biais and Green (2019) show that this migration happened as composition of secondary debt markets changed and institutional investors came to account for the majority of the trading activity. They conjecture that these large investors might have found the opacity of OTC dealer markets preferable to the transparency of limit order markets. Consistent with this, anonymity has been front and center in the debate about the modernization of corporate bond markets microstructure. Investors have favored all-to-all electronic venues, where counterparties can trade anonymously with one another (https://bloomberg.com/news/articles/2019-04-01/wall-street-is-getting-cut-out-of-bond-market-it-long-dominated), and limiting post-trade reporting of large block trades to minimize the price impact of such orders (https://www.blackrock.com/corporate/literature/whitepaper/viewpoint-addressing-market-liquidity-july-2015.pdf).

Greenwich Associates\textsuperscript{6}, the vast majority of the trades in secondary bond markets (over 90\% of trades of $100k or less) are now done in electronic platforms. Nonetheless, trades of larger ticket size ($1MM or more), which make up over 80\% of the notional volume traded daily, are still done over-the-counter. The main obstacle to electronic trading of corporate bonds is arguably the fragmentation of trading activity across a vast universe of securities in this asset class. Unlike in equity markets, where large companies issue at most a few dozen stocks, in corporate bond markets large issuers can have from a few hundred to thousands of bonds outstanding, the vast majority of which trades only infrequently. While multiple and varied issuances allow companies to minimize debt refinancing risks by diversifying their capital structure and debt maturity schedule, they also render each security individually more illiquid. Put differently, with such a large number of bonds available to trade, it is practically impossible for an investor to find a natural counterparty to a trade at any particular time.

To address market fragmentation, banks and large investment firms such as BlackRock have been pushing for the standardization of corporate bonds. Standardization can improve liquidity by facilitating pricing and by broadening the pool of potential investors. In addition, it allows for the development and integration of trading and settlement systems. For example, new trade protocols are being developed to enable multiple counterparties to simultaneously fulfill pieces of a single large order, increasing market depth by reducing the execution time of block trades.\textsuperscript{7} Regulatory agencies have observed that, by promoting more centralized trading of bonds, standardization can help reduce systemic risks as trade execution risks migrate from dealers to end-investors in fixed-income markets.\textsuperscript{8} Finally, standardization of corporate bonds makes it easier for the industry to adopt standardized index products, such as ETFs, and hedging tools, such as interest rate swaps and credit default swaps, which in turn increase the liquidity of the underlying bonds.

\subsection*{B. The Informational Role of Bond Covenants}

Standardization can be fairly straightforward in some more homogenous asset classes. However, corporate bonds own legal and financial idiosyncrasies have always been an obstacle. Traditionally, most clients in the dealer-customer segment are buy-and-hold institutional investors, who often time look for securities specially tailored to their need for exposure to certain risks. For this reason, bonds often include contractual clauses designed to shield investors from losses. These clauses can work by precluding firms from acting in a way that is detrimental to creditors, or ensuring repayment in case of certain contingencies.

A bond covenant is a provision, such as a limitation on the payment of dividends, which restricts the firm from engaging in specified actions after the bonds are sold. \textit{(Smith and Warner (1979))}

Examples of such covenants include \textit{make-whole} redemption compensations, which guarantee a lump sum payment to bond investors in case the debt is called off before maturity to compensate them for the foregone coupon payments; and restrictive covenants preventing merger activities or the issuance of new debt. Other restrictive covenants preclude companies from paying dividends to shareholders after missing an interest payment to bond investors, or from selling the firms’ assets. Such clauses are designed to protect bondholders from the payout of assets pledged as collateral.

\textsuperscript{6}https://www.greenwich.com/blog/challenge-trading-corporate-bonds-electronically

\textsuperscript{7}See footnote 6.

A push to increase liquidity by homogenizing corporate bonds traded in electronic platforms might do away with creditor protection clauses, leaving investors exposed to certain firm-specific risks that are difficult to hedge. On the other hand, the benefits of electronization might be hindered if market participants deem covenants too important to be standardized or outright eliminated. In this case, a hybrid market structure might prevail, where more complex, non-standardized debt trades over-the-counter, whereas covenant-lite bonds are actively transacted in electronic venues.

The next section presents the core theoretical framework of the analysis, namely the types of investors, the micro-structure of secondary markets, the debt and equity valuation formulas, the choice of capital structure, and firms’ endogenous bankruptcy decision. Section III introduces an asymmetry of information between creditors and shareholders and discusses the ensuing conflict of interest that can arise between these two classes of investors. The following section explains debt standardization and distinguishes between over-the-counter and electronic markets. Section V derives the equilibria when the only secondary market is the electronic platform, and discusses their properties. Next, the analysis is extended to accommodate both over-the-counter and electronic markets. Section VII concludes.

II. The Structural Credit Risk Model

To investigate the consequences of bond standardization over firms’ choice of capital structure and the composition of debt across secondary markets, I propose a structural model of credit risk with asymmetric information, where debt protective covenants arise endogenously. In the model, covenants constitute a costly way to produce private information and mitigate bondholder-shareholder conflicts in electronic platforms.

The model features two classes of investors, bond investors (or creditors) and equity holders (shareholders), and two competing secondary markets, over-the-counter (OTC) markets and an electronic platforms (EPs.) Secondary markets differ in (i) their (external) liquidity and (ii) the types of bonds they accept. Electronic exchanges offer lower transaction costs, but intermediate only trades of standardized, covenant-free bonds, whereas OTC markets accept any type of bond.

Equity investors observe investment opportunities (or projects), which can be either safe or risky, depending on their exposure to an idiosyncratic, unhedgeable risk. To invest in projects, these investors set up firms, which are financed with a mix of finite-maturity bonds and shares. Debt allows firms to benefit from tax-shields, but introduce the risk of costly bankruptcy. Each firm carries out one and only one project and commits to a fixed capital structure, defined by the type and measure of outstanding bonds at any given time. Finally, creditors require firms to choose their debt instruments and leverage to maximize their initial valuation.

All else constant, debt issued with standardized, covenant-free bonds is more valuable because these bonds are traded in the more liquid electronic markets. The reduced transaction costs of secondary trades in EPs render newly-issued bonds more valuable, thereby lowering firms’ debt rollover costs. Absent any asymmetry of information, therefore, all firms issue standardized bonds, and their levered capital structures reflect their exposure to the unhedgeable risky.

When bond investors are not fully informed about the firms, however, risky firms’ shareholders may be able to increase their rate of return by misrepresenting their firms’ type. Depending on (i) the ratio of safe to risky firms, and (ii) the risky type’s exposure to the unhedgeable risk, misrepresentation can benefit risky-type shareholders by allowing their firms to copy the more levered capital structure of safe firms and enjoy relatively lower debt rollover costs. Type misrepresentation is thus akin to an asset substitution problem, wherein creditors’ valuation of a firm’s debt is incommensurate with the firm’s riskiness.
While bond investors may not observe firms’ underlying exposure to the unhedgeable risk, their knowledge of the firm-type distribution allows them to anticipate each type’s strategies. Therefore, misrepresentation prompts creditors to revise downwards their valuation of all standardized bonds, thereby raising the debt rollover costs for safe firms. Safe firms in turn adjust their debt-equity ratio, either by increasing their measure of outstanding bonds to discourage the risky type’s misrepresentation, or by reducing their leverage to minimize the impact of the misrepresentation over their debt-rollover costs. Alternatively, safe firms may issue bonds with a debt protective covenant to signal their creditworthiness.

I decompose the safe type’s total return differential under full and asymmetric information into a liquidity and an informational components. The liquidity component is a function of the difference in market-specific transaction costs. The informational component measures the cost of adverse selection to the absolute return of safe projects in electronic markets, and is a function of the distribution of types. More specifically, informational costs decrease with the measure of safe firms, and rise with the risky type’s exposure to the unhedgeable shock.

When the unhedgeable risk differential between the two types of firms is small or the ratio of safe-to-risky firms is sufficiently high, the informational cost of adverse selection in electronic platforms is minimized by having safe firms reduce their leverage so that types pool together. When the risky differential is large or the measure of safe firms is small, however, the effect of pooling over the safe firms’ debt rollover costs is so high that these firms find it preferable to increase their leverage to discourage the risky type’s misrepresentation.

Covenants arise endogenously as a means of maximizing the absolute returns of safe projects by eliminating the informational asymmetry between the different types of investors. When informational costs exceed the liquidity differential between over-the-counter and electronic markets, safe firms forego the liquidity gains and issue instead non-standardized bonds to signal their creditworthiness. In this case, a dual-market separating equilibrium holds where only risky firms issue standardized bonds.

I now describe the market structure more precisely. I assume there exists a risk-neutral measure under which all assets in the economy are priced. Therefore, henceforth all stochastic processes are defined against this measure.

### A. Firms

Firms are economic entities set up by equity holders to invest in a project and require an initial capital injection of $V_0$. From then on, the value of a firm’s underlying assets $\{V_t: 0 \leq t < \infty\}$, also referred to as the **fundamental value of the firm**, evolves according to

$$\frac{dV_t}{V_t} = (r - \delta) dt + \sigma_t dZ_t$$

where $r$ is the risk-free rate, $\delta$ is the gross dividend payout rate, $\sigma_t$ is the underlying assets’ volatility, and $\{Z_t: 0 \leq t < \infty\}$ is a standard Brownian motion under the risk-neutral measure. Each firm handles a single project, so I apply the terms safe and risky to firms and projects interchangeably.

A project’s riskiness is captured by the parameter $\sigma$ above. All firms start out with low volatility $\sigma_i$, but differ in their exposure to a volatility shock, modeled as a Poisson process of intensity $\lambda$.\(^9\)

Upon the arrival of the first shock, a firm’s asset volatility jumps permanently to $\sigma_h$, for $\sigma_h > \sigma_l$.

\(^9\)The Poisson distribution is convenient because of its memorylessness property, which ensures the probability of a shock is not path-dependent. At every instant $dt$, the firm faces a constant probability $\lambda \cdot dt$ of becoming riskier. This allows me to restrict the set of state variables to the firm’s asset value, $V$, and volatility, $\sigma$. 


so that
\[ \sigma_t = \sigma_l + (\sigma_h - \sigma_l) \cdot 1_{\{t \geq t^d\}}, \quad \sigma_h > \sigma_l \]

where \( t^d \) is the first-passage time of the volatility shock process and \( 1_{\{\cdot\}} \) is the indicator function. Types are then fully characterized by their exposure to the volatility shock, consisting in the pair of the volatility shock intensity, \( \lambda \), and post-shock volatility level, \( \sigma_h \).

**ASSUMPTION 1:** *(Firm Types)* All firms pay dividends at rate \( \delta \) and start with the same volatility, \( \sigma_l \), and underlying asset value, \( V_0 \). Firm types differ only in their exposure to the volatility risk \((\lambda, \sigma_h)\), and are therefore fully characterized by a pair \( \gamma \), so that
\[ \gamma \in \{(\lambda_s, \sigma_{h,s}), (\lambda_r, \sigma_{h,r})\} \]
where subscripts \( s \) and \( r \) stand for safe and risky types, respectively.

To facilitate the analysis, I assume only risky types are exposed to the volatility risk, so that \( \gamma_s = (0, \sigma_l) \) and \( \gamma_r = (\lambda, \sigma_h) \), with \( \lambda > 0 \) and \( \sigma_h > \sigma_l \). Each firm operates indefinitely, until \( V_t \) falls to an endogenously determined default-triggering value \( V^B \). At this point, a firm is declared bankrupt. The dependency of the default barrier on the model parameters, and more specifically on the firm’s volatility, will be explained later.

### A.1. Firm Cohorts

A firm cohort is comprised of all firms that enter the economy at a specific time \( t \geq 0 \). In each cohort, firm types are independently drawn from the same exogenous and time-invariant Bernoulli distribution. Without loss of generality, I normalize the measure of entrant firms to 1 and denote by \( \mu_s \) the probability of an entrant firm being of the safe type, so that:

**ASSUMPTION 2:** *(The Distribution of Firms)* The distribution of firms is exogenous, time-invariant and known by bond investors. Without loss of generality, I normalize the measure of firms to 1, so that
\[ \mu_s \equiv \mu_t (\gamma_s) = 1 - \mu_t (\gamma_r) \]
for all \( t \geq 0 \).

Let \( \gamma \equiv (\gamma_s, \gamma_r) \). Taken together, \( \gamma \) and \( \mu_s \) fully characterize the type-distribution of entrant firms in any cohort.

### A.2. Stationary Debt Structure

Firms can be financed by a mix of equity and debt. Debt issuance allows them to benefit from tax shield incentives, but exposes investors to the risk of a costly bankruptcy process, as equity holders cannot be made liable for losses that exceed the value of the firm. Debt is raised via the issuance of a single type of debt instrument: a bond of maturity \( m \), annual coupon \( c \) and annual principal \( p \). I refer to the tuple \( b \equiv (m, c, p) \) as a *bond contract*. For now, I consider only covenant-free bonds. Later on, I show that debt protective covenants arise endogenously as a means of mitigating agency costs.

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10Together, \( \lambda \) and \( \sigma_h \) determine a firm’s overall exposure to the volatility shock. In section B of the Online Appendix, I show these parameters have similar qualitative effects over the values of equity and debt and the firms’ bankruptcy barrier. Therefore, the volatility-risk exposure can in principle be reduced to a one-dimensional variable. For example, the effect of any pair \((\lambda_r, \sigma_{h,r})\) over a risky firm’s choices of risk-management, capital structure and default can be proxied by \((\lambda^*, \sigma^*_{h,r})\), for some \( \lambda^* > \lambda_r \) and \( \sigma^*_{h,r} < \sigma_{h,r} \).
As in Leland and Toft (1996) and He and Xiong (2012), firms commit to a stationary debt structure at their inception. This structure consists in a continuum of bonds of the same seniority and bond contract \( b \), differing only in their maturity dates. Firms’ debt maturity profile is kept constant and uniformly distributed across time by the continuous issuance of bonds with identical characteristics to replace maturing ones. Letting \( \mu_b \) denote the constant density of outstanding bonds of initial maturity \( m \), a firm’s debt structure is fully characterized by the vector \( \omega \equiv (\mu_b, b) \).

**ASSUMPTION 3:** A firm’s capital structure is chosen at inception and kept fixed for as long as the firm is alive. This structure consists of a measure \( \mu_b \) of bonds of the same contract \( b \equiv (m, c, p) \), differing only in their time-to-maturity.

Finally, the aggregate levels of annual coupon and principal payments are computed as follows

\[
C = \int_0^m c \cdot \mu_b(\tau) \, d\tau = c \cdot m \cdot \mu_b
\]

and

\[
P = \int_0^m p \cdot \mu_b(\tau) \, d\tau = p \cdot m \cdot \mu_b
\]

where \( m \cdot \mu_b \) is the measure of outstanding bonds at any given time.

### A.3. Debt Rollover and Endogenous Bankruptcy

Cash-flows happen on a continuous basis and all net cash-flows immediately accrue to equity holders. Since, by assumption, all operating expenses are captured by the stochastic process for the fundamental value of the firm, \( \{V_t\}_{t=0}^{\infty} \) in equation 1, for the purposes of firm valuation it suffices to determine the firm’s cash payout (dividends) and debt-related cash-flows.

The debt-related cash-flows consist on the firm’s expenses servicing the debt and the proceedings from new debt issuance. The stationarity of firms’ capital structure discussed above ensures that, in any interval of time \( dt \), interest due on outstanding debt and the principal repayment on maturing bonds sum-up to \((1 - \pi) C \cdot dt + p\), where \( \pi \) denotes the marginal tax benefit of debt. Firms receive \( d(V_t, m) \) for each newly-issued bond. The difference between the funds raised by the issuance of new debt and the repayment of bond principal, \( d(V_t, m) - p \), is called the debt rollover profit/loss.

Over a short interval \((t, t + dt)\), equity holders receive dividend payments of the order of \( \delta V_t \cdot dt \), and must cover the coupon and debt rollover expenses. Therefore, the net cash-flow to equity holders (omitting \( dt \)) is

\[
NC_t = \delta V_t - (1 - \pi) C + d(V_t, m, \sigma_t) - p
\]

Debt Rollover profit/loss (4)

Shareholder demand is perfectly elastic so long as the value of equity is positive. When net cash-flows are negative, losses are paid off via the issuance of more equity at market prices, a process known as “equity dilution”. Default is then determined endogenously, when the value of equity reaches zero.

The frictionless equity market assumption and the cash-flow based valuation approach make the model particularly suited to the study of the effects of a liquidity differential across secondary debt markets. As in He and Xiong (2012), the firm value is governed solely by the underlying asset value and the expected future rollover gains/losses. As will be shown, rollover costs are affected by liquidity conditions faced by bond investors, which feedback into the firms choice’ of leverage and bankruptcy at time 0.
A.4. Investors’ Intra-period Timing

The timing of events within a period $t \geq 0$ for investment in an entrant firm is as follows. At the beginning of the period, shareholders observe investment opportunities drawn from the time-invariant Bernoulli distribution characterized by $(\gamma, \mu_s)$. For each project they decide to invest in, these investors set up a firm. Entran t firms then commit to a stationary capital structure, $\omega \equiv (\mu_b, b)$, and issue debt and equity shares. This intra-period timeline is represented in figure 1 below.

![Figure 1. Investors’ Intra-period Timing for Investment in Entrant Firms](image)

The figure depicts the sequence of actions within a period $t \geq 0$ for investment in an entrant firm.

The timing of events for established (non-entrant) firms involves the monitoring of the occurrence of the volatility shocks. At the beginning of a period $t > 0$, all investors observe whether a volatility shock has occurred. Next, the firm’s cash-flows are computed. If the value of equity falls to zero, the firm is declared bankrupt, and its assets are liquidated to repay outstanding debt. Else, its debt is rolled over. Finally, the net cash-flows are realized. The schematic is depicted in figure 2.

![Figure 2. Investors’ Intra-period Timing for Investment in Established Firms](image)

The figure depicts the sequence of actions within a period $t \geq 0$ for investment in an established firm.

B. Secondary Bond Markets

I restrict the source of illiquidity affecting firms’ financing decisions to the microstructure of secondary debt markets. As in He and Xiong (2012), I assume (i) the demand for bonds is infinitely elastic, (ii) trade is governed by a stochastic process and (iii) illiquidity is proxied by an exogenous portfolio liquidation cost.

Investors can absorb any quantity of bonds at market prices, but are subject to liquidity shocks. Shocks are idiosyncratic and arrive according to independent Poisson processes with intensity $\xi$. 
Upon the arrival of a shock, a bond investor must immediately liquidate her position at a market-specific, fractional cost $\kappa$. That is to say, bondholders recover only a fraction $1 - \kappa$ of the bond price, where the value $\kappa$ can vary from one secondary trading venue to another.

The transaction cost parameter proxies for market frictions that affect the execution time and price of a trade and that ultimately lead investors to accept a discount on the value of their positions. This formulation is consistent with a market structure where trades in secondary markets are intermediated by competitive market makers who charge a spread to compensate them for absorbing transitory excess demand or supply in their inventory positions. As I show in the next section, the risk of a costly liquidation increases bond investors’ rate of discount, which negatively affects the price of newly issued bonds.

All else constant, given competing choices of trading venues, investors will choose the more liquid one. But limitations to the type of debt instruments transacted in each venue can affect firms’ choice of leverage and the debt composition across markets. For now, I assume there is only one type of secondary market. Later on I will show how the model can be used to study the implications of the rise in electronic trading for over-the-counter markets.

C. Debt Valuation when creditors observe a firm’s type

Because by assumption credit markets are perfectly competitive and firms commit to a stationary capital structure $\omega$, at any time $t \geq 0$ the state of a firm can be fully captured by 3 state variables: (i) the value of its underlying assets, $V_t$, (ii) the firm’s type, $\gamma_j$ for $j \in \{s, r\}$, and (iii) an indicator of whether the firm has experienced a volatility shock, $1_{\{\sigma = \sigma_h\}}$. I assume, and later on prove, that the optimal bankruptcy policy in this setting consists in declaring default the first time the value of the firm’s underlying assets crosses a fixed threshold from above for the first time.

In this section, I derive the value of debt for an arbitrary bankruptcy barrier, $V^B \in (0, 1)$. I begin by pricing the bonds of a safe firm. As discussed above, the volatility of a safe-type firm is always constant. Risky-type firms have constant volatility only after the arrival of their first idiosyncratic shock. I refer to these firms as post-volatility-shock firms.

The value of a bond with maturity and cash-flows given by $b$, when the fundamental value of the firm is $V_t$ and the firm’s default barrier is $V^B$, is $d_\sigma (V_t, \tau; V^B, \omega, \theta)$, where subscript $\sigma$ indicates that volatility is fixed. By Ito’s Lemma, this pricing function must satisfy the following partial differential equation (5):

$$r_{disc} \cdot d_\sigma (V_t, \tau; V^B, \omega, \theta) = c + \frac{\partial}{\partial t} d_\sigma (V_t, \tau; V^B, \omega, \theta) + r_{grow} \cdot V_t \frac{\partial}{\partial V} d_\sigma (V_t, \tau; V^B, \omega, \theta)$$

$$+ \frac{1}{2} \sigma^2 V_t^2 \frac{\partial^2}{\partial V^2} d_\sigma (V_t, \tau; V^B, \omega, \theta)$$

with $r_{grow} = r - \bar{\gamma}$, $r_{disc} = r + \xi \kappa$ and $\theta \equiv (r_{grow}, r_{disc}, \sigma)$.

A direct application of the Feynman–Kac Theorem to the PDE above shows that the investors’ de facto discount rate, $r_{disc}$, equals the sum of the risk-free interest rate, $r$, and a strictly positive liquidity risk premium term, $\xi \kappa$. This premium arises from the costly forced liquidation discussed above. In the online appendix A, I offer a more direct, and perhaps more intuitive, derivation of $r_{disc}$ in a more general setting that allows for arbitrary cash-flows.

The bond pays the annual coupon up until maturity or default, whichever comes sooner. When maturity happens before default, a bond holder receives the principal value:

$$d_\sigma (V_t, 0; V^B, \omega, \theta) = p, \text{ for all } V_t > V^B$$

\[11\] A derivation can be found in the online appendix, available here: https://abcarvalho.github.io/files/abcarvalho_online_app.pdf
Conversely, when the firm is declared bankrupt, all outstanding debt comes due and the firm’s assets are liquidated at a fractional cost \((1 - \alpha)\), \(\alpha \in (0, 1)\), to repay investors. Debt claims have priority over equity shares (“absolute priority” rule), so the recovery value on the outstanding debt is proportional to the firm’s fundamental value and equals \(\alpha V^B\).\(^{12}\) For simplicity, I impose the pari-passu clause, under which all bonds have the same seniority, regardless of their time-to-maturity. Therefore, each bond pays an equal share of the firm’s liquidation value upon default

\[
d_{\tau} (V^B, \tau; V^B, \omega, \theta) = \frac{\alpha V^B}{\mu_b \cdot m}, \quad \text{for all } \tau \in [0, m] \tag{7}
\]

The value of a defaultable bond with time-to-maturity \(\tau\) when the volatility of the firm is constant is

\[
d_{\tau} (V_t, \tau; V^B, \omega, \theta) = \frac{c}{r_{\text{disc}}^\tau} + e^{-r_{\text{disc}}^\tau} \left[ p - \frac{c}{r_{\text{disc}}^\tau} \right] (1 - F(\tau, v_t; \theta_0)) + \left[ \frac{\alpha V^B}{\mu_b \cdot m} - \frac{c}{r_{\text{disc}}^\tau} \right] G(\tau, v_t; \theta) \tag{8}
\]

where \(v_t = \ln (V/V^B)\), \(\omega = (\mu_b, b)\), \(\theta_0 = (r_{\text{grow}}, \sigma)\), \(\theta = (r_{\text{disc}}, r_{\text{grow}}, \sigma)\), \(r_{\text{grow}} = r - \delta\), \(r_{\text{disc}} = r - \xi \kappa\) and the functions \(F(\cdot)\) and \(G(\cdot)\) are as defined in Theorem 1 in Appendix E.\(^{13}\)

Equation 8 can be decomposed into the values of more primitive claims. The term \(e^{-r_{\text{disc}}^\tau} (1 - F(\tau, v_t; \theta_0))\) is the value of a claim that pays \$1\) at maturity if the firm does not declare bankruptcy, while \(G(u, v_t; \theta)\) stands for the price of a claim that pays \$1\) upon default if default happens before maturity. Therefore, \(d_{\tau} (V_t, \tau; V^B, \omega, \theta)\) can be interpreted as the value of a claim that pays a coupon \(c\) up until maturity or default, whichever comes first

\[
\frac{c}{r_{\text{disc}}^\tau} \left[ 1 - e^{-r_{\text{disc}}^\tau} (1 - F(\tau, v_t; \theta_0)) - G(u, v_t; \theta) \right]
\]

plus the value of a claim that pays \(p\) at maturity if the firm is not bankrupt

\[
p \times e^{-r_{\text{disc}}^\tau} (1 - F(\tau, v_t; \theta_0))
\]

\(^{12}\)The book value of a firm’s debt is

\[
D_{\text{crf}} (r_{\text{disc}}, \omega) = \int_0^m b_{\text{crf}} (\tau; r_{\text{disc}}, \omega) \cdot \mu_b \cdot d\tau
\]

where \(b_{\text{crf}} (\tau; r_{\text{disc}}, \omega)\) is the book value of a bond with maturity \(\tau\), when the investor’s rate of discount is \(r_{\text{disc}}\) and the bond’s maturity, coupon and principal satisfy the capital structure \(\omega\):

\[
b_{\text{crf}} (\tau; r_{\text{disc}}, \omega) = \int_{s=0}^{\tau+\tau} e^{-r_{\text{disc}}^s} (s-\tau) + e^{-r_{\text{disc}}^\tau} p = \frac{c}{r_{\text{disc}}} + e^{-r_{\text{disc}}^\tau} \left[ p - \frac{c}{r_{\text{disc}}} \right]
\]

and superscript \(\text{crf}\) stands for “credit-risk-free”.

The recovery value on the firm’s debt is the minimum of the liquidation value of the firm, \(\alpha V^B\), and the book value of debt.

\[
\text{Rec}(\tau) = \min \left( \alpha V^B, D_{\text{crf}} (r_{\text{disc}}, \omega) \right)
\]

Suppose, by way of contradiction, that the liquidation value of the firm is greater than the amount it owes creditors. Since \(b_{\text{crf}} (\tau; r_{\text{disc}}, \omega) \geq d_{\tau} (V^B, \tau; V^B, \omega, \theta)\) for all \(\tau \in [0, m]\) and \(V_t \geq 0\), the book value must be greater than the market value of the firm’s debt. Therefore,

\[
\alpha V^B > D_{\text{crf}} (r_{\text{disc}}, \omega) \geq \int_0^m d_{\tau} (V_t; \tau; V^B, \omega, \theta) \cdot \mu_b \cdot d\tau, \quad V \geq V^B
\]

which can only happen if the firm is declared bankrupt when the value of equity is strictly positive, a contradiction!\(^{13}\) This pricing function is identical to that in He and Xiong (2012) when the measure of outstanding bonds is normalized to 1.
and the value of a claim that pays $\frac{\alpha V_B}{\mu_b \cdot m}$ upon default if bankruptcy happens before maturity:

$$\frac{\alpha V_B}{\mu_b \cdot m} \times G(\tau, v_t; \theta)$$

The pricing formula for the bond of a risky firm is a bit more involved. By assumption, the high-volatility is an absorbing state. Once a pre-volatility-shock firm receives a shock, its volatility is permanently increased to $\sigma_h$. At this point, outstanding bonds are priced by the constant-volatility formula above. Even though the firm’s capital structure is fixed, its bankruptcy condition is adjusted to reflect the shift in the dynamics of the underlying assets and the accompanying changes in debt rollover costs. Denote by $V_{t} \, B^{l}$ and $V_{h}^{B}$ the pre- and post-volatility-shock bankruptcy barriers, respectively. The bond price is then

$$d \left( V_t, \tau; V^B, \lambda, \omega, \theta \right) = \frac{c}{r_{disc} + \lambda} + e^{-\left( r_{disc} + \lambda \right) \tau} \left[ p - \frac{c}{r_{disc} + \lambda} \right] \left[ 1 - F \left( \tau, v_t; \sigma_l \right) \right]$$

$$+ \left[ \frac{\alpha V_t}{\mu_b \cdot m} - \frac{c}{r_{disc} + \lambda} \right] G(\tau, v_t; \theta_l)$$

$$+ \lambda \int_{t}^{\tau+\tau} e^{-\left( r_{disc} + \lambda \right) \left( s - t \right)} \left\{ \int_{V_t}^{\infty} d\pi \left( V, \tau - s; V_{h}^{B}, \omega, \theta_h \right) \times \right.$$  

$$\left. \times \left[ -\frac{\partial}{\partial V} \Psi^{v,t^{d}} \left( s - t, v_t; \sigma_l \right) \right] dV \right\} ds \]$$

where $V^B \equiv (V^B_t, V^B_h)$, $v \equiv \ln(V/V^B_t)$, $\theta \equiv (r_{grow}, r_{disc}, \sigma_l, \sigma_h)$, $\theta_j \equiv (r_{disc}, r_{grow}, \sigma_j)$, $j = l, h$, $r_{grow} = r - \delta$ and $r_{disc} = r - \xi \kappa$. The functions $F(\cdot), G(\cdot), \Psi^{v,t^{d}}(\cdot)$ are defined in Theorem 2 in Appendix E and $d\pi(\cdot)$ is the constant-volatility bond price in Theorem 1.

The first two lines on the RHS are analogous to the constant-volatility bond pricing formula in Theorem 1, except for the volatility-shock hazard rate $\lambda$. The volatility-shock-adjusted discount rate, $r + \lambda$, accounts for the possibility of a liquidity shock happening before maturity. The terms can be interpreted as a claim to a defaultable bond that pays off up to maturity, default or liquidity shock, whichever happens first. Finally, the double integral captures the bond’s expected payoff conditional on the volatility shock arriving before maturity. I compute this last term numerically.\(^{14}\)

The value of debt is obtained by integrating the bond price function with respect to the measure of outstanding bonds.

$$D \left( V_t; V^B, \omega, \theta \right) = \mu_b \cdot \int_{0}^{m} d \left( V_t, \tau; V^B, \omega, \theta \right) d\tau$$

\(D.\) **Equity Valuation when creditors observe a firm’s type**

Illiquidity in secondary bond markets reduces the value of the levered firm that accrues to shareholders. As explained in the previous section, transaction costs introduce a wedge between the market’s and the firm’s valuations of debt cash-flows. This differential is reflected in the liquidity premium term on investors’ rate of discount, $\xi \kappa$.

Pricing the effect of future transaction is complicated because they affect the firm’s net cash-flows in a non-linear way, as it can be seen from the bond pricing formula in Theorem 1. Therefore, when volatility is constant, I follow He and Xiong (2012) and compute the value of equity by directly

\(^{14}\) For a detailed explanation of the numerical approach, see section C in the online appendix.
solving its partial differential equation:

\[
    r E^\sigma (V_i; \omega, \theta) = r_{\text{grow}} V_i E^\sigma (V_i; \omega, \theta) + \frac{1}{2} \sigma^2 V_i^2 E_{VV}^\sigma (V_i; \omega, \theta) \\
    + \delta V_i - (1 - \pi) C + d\pi (V_i, m; V^B_i, \omega, \theta) - p
\]

(11)

The derivation of the expression above and its solution can be found in Appendix F. The optimal default barrier, \(V^B(\omega, \theta)\), is obtained by imposing the limited liability constraint and requiring that the value of equity converges to zero as the firm approaches bankruptcy (smooth-pasting condition.) The bankruptcy formula can be found in Theorem 4 in Appendix F and is identical to that of Proposition 1 in He and Xiong (2012) when \(\mu_b = 1\).

As conjectured, \(V^B\) is independent from the underlying value of assets and constant. At any level below this threshold, the value of equity becomes negative. Therefore, equity dilution is no longer a viable option to fund the service of debt. Keeping the firm alive beyond this point would require losses to be covered with external cash. Because shareholders cannot be made liable for the losses of bond investors, they choose optimally to default when \(V\) reaches \(V^B(\omega, \theta)\).

I now consider the case when volatility is stochastic. For an arbitrary default barrier \(V^B_t\), denote by \(E(V; V^B_t, \lambda, \omega, \theta)\) the equity value function of a pre-volatility-shock firm. I show in Appendix G.1 that this function must satisfy the following PDE:

\[
    (r + \lambda) E (V_t; V^B_t, \lambda, \omega, \theta_t) = \delta V_t - (1 - \pi) C + d (V_t, m; V^B_t, \lambda, \omega, \theta_t) - p + \lambda E^\sigma (V; \omega, \theta_h) \\
    + (r - \delta) V_t \frac{\partial}{\partial V} E (V_t; V^B_t, \lambda, \omega, \theta_t) + \frac{1}{2} \sigma^2 V_t^2 \frac{\partial^2}{\partial V^2} E (V_t; V^B_t, \lambda, \omega, \theta_t)
\]

(12)

where \(d (V_t, m; V^B_t, \lambda, \theta_t)\) is the pre-volatility-shock bond price in equation 9 above and \(E^\sigma (\cdot; \omega, \theta_h)\) is the equity function in equation 11, when \(\sigma = \sigma_h\) and the post-volatility-shock bankruptcy barrier is set optimally to \(V^B(\omega, \theta_h)\), according to Theorem 4 in Appendix F.

The PDE above is solved numerically via the Finite Differences Method, as outlined in Appendix G.2. By construction, the value of the equity function at the lower boundary is set to zero. To identify the optimal bankruptcy barrier then, I search for the bankruptcy value \(V^B_t\) satisfying

\[
    \lim_{V^B_t \to V^B_t} \frac{\partial}{\partial V} E (V; V^B_t, \lambda, \omega, \theta_t) = 0 \quad \text{(smooth-pasting)}
\]

(13)

\[
    \frac{\partial}{\partial V} E (V; V^B_t, \lambda, \omega, \theta_t) \geq 0, \quad \forall V \geq V^B_t
\]

(14)

The first equation is simply the smooth-pasting condition. Combined with the limited liability condition, it states that the value of equity should converge smoothly to zero as the firm approaches bankruptcy. Inequality 14 is designed to avoid an overshooting.\(^{15}\)

E. The Choice of the Capital Structure

I now discuss the choice of leverage. I assume firms are under the direct control of the shareholders, but subject to creditor oversight. Before entering the economy, firms commit to a stationary

\(^{15}\)For low enough \(V^B_t\) values, the numeric approach can return a function that becomes negative and changes concavity as \(V \to V^B_t\). In such cases, the function may satisfy equation 13, while violating the limited liability condition.
capital structure, \( \omega \). The total risk-adjusted return to shareholders at the time a firm enters the market is the difference between the equity value, \( E_0 \), and shareholders’ net cash infusion, \( V_0 - D_0 \):

\[
TR_0 (V_0; \lambda, \omega, \theta) = E_0 (V_0; V_B (\lambda, \omega, \theta), \lambda, \omega, \theta) - \left[ V_0 - D (V_0; V_B (\lambda, \omega, \theta), \omega, \theta) \right]
\]

\[
= \left[ E_0 (V_0; V_B (\lambda, \omega, \theta), \lambda, \omega, \theta) + D (V_0; V_B (\lambda, \omega, \theta), \omega, \theta) \right] - V_0
\]

where \( V_B (\lambda, \omega, \theta) \) is the optimal default barrier. Thus, shareholder’s total return is maximized by choosing the capital structure \( \omega \) that yields the highest initial firm valuation. The corresponding risk-adjusted rate of return is the market-to-book ratio of equity (MBR), defined as the ratio of the market the value of equity to the cash amount put down by equity holders at inception.

\[
MBR (V_0; \lambda, \omega, \theta) \equiv \frac{E (V_0; V_B (\lambda, \omega, \theta), \lambda, \omega, \theta)}{V_0 - D (V_0; V_B (\lambda, \omega, \theta), \omega, \theta)} - 1
\]  

(15)

As will be discussed in the next section, maximization of the initial firm value does not coincide with the maximization of shareholder’s rate of return. I assume that, to the extent of their ability to differentiate between project types, creditors enforce the capital structure \( \omega \) that maximizes the total economic value of firms’ underlying assets by refusing to fund projects that are set up with a different leverage. In a full information setting then, creditor oversight effectively transfers control of the firms from shareholders to debt investors.

ASSUMPTION 4: \{Creditors’ Funding Condition\} When creditors’ fully observe a firm’s type and actions, they require that the choice of capital structure be made to maximize the total firm value (debt + equity).

By assumptions 3 and 4, the firm chooses a stationary capital structure \( \omega \) at time 0 satisfying

\[
\omega^* (\lambda, \theta) \in \arg \max_{\omega \in \mathbb{R}^+_4} \left\{ D (V_0; V_B (\lambda, \omega, \theta), \omega, \theta) + E (V_0; V_B (\lambda, \omega, \theta), \lambda, \omega, \theta) \right\}
\]  

(16)

where \( D (\cdot) \) and \( E (\cdot) \) are the constant- or stochastic-volatility debt and equity functions, depending on whether the firm is safe or risky. In either case, the bankruptcy barrier \( V_B (\lambda, \omega, \theta) \) is optimally set to satisfy the limited liability constraint.

Notice there can be multiple capital structures satisfying condition 16 above. Indeed, when the firm is free to set the coupon ratio \( c/p \) as well as the measure of bonds \( \mu_b \), this first optimization step yields infinite solutions. Without loss of generality, I normalize the measure of bonds to 1 and require that the principal \( p \) be such that debt is issued at par.\(^{16}\) Finally, \( \omega^* \) is determined by setting the bond coupon to the value \( c > 0 \) that yields the first local maxima of the firm value function 16.

I call optimal payoffs the payoffs of the firm-value-maximizing capital structure \( \omega^* \). Firms then enter the market at time-0 if the shareholders’ optimal payoff, equity’s market-to-book ratio \( MBR (V_0; \lambda, \omega^* (\lambda, \theta), \theta) \), is strictly positive.

DEFINITION 1: Optimal payoffs: payoffs obtained when the capital structure is set to \( \omega^* (\lambda, \theta) \).

For simplicity, I keep maturity fixed at 1 year throughout the analysis, unless otherwise noted. The shock intensity and the shock size yield similar qualitative effects on firms’ choices: an increase in

\(^{16}\)The algorithm to solve the first optimization step is discussed in Appendix H.
either component of the volatility-risk exposure decreases risky firms’ optimal leverage, bankruptcy barrier and total value, as well as the optimal market-to-book ratio of equity.\(^\text{17}\)

III. Private Information and Agency Costs

Up until now, the model assumed bond investors had perfect knowledge of the riskiness of firms. In reality, however, credit markets are notoriously opaque. This opacity hinders the price discovery process and can lead to adverse selection. I now show that, when creditors are less informed than shareholders, the misalignment between the economic value of a firm’s assets and investors’ payoffs can be exploited by equity holders.

A. Investors’ Conflicting Interests and Asymmetric Information

The optimality of a firm’s capital structure typically does not coincide with the maximization of investors’ expected payoffs. In fact, the MBR can be arbitrarily inflated by increasing a firm’s leverage. In a setting where creditors fully observe the risks of a project, the funding condition (assumption 4) effectively constrains shareholders’ choice of the debt-equity mix. But, when firm-specific information is made privy to a specific group of investors, the misalignment between the economic value of a firm’s assets and investors’ payoffs can lead to agency problems. Consider the case where debt holders do not directly observe the firms’ types.

ASSUMPTION 5: [Public Information] The time-invariant entrant firm type distribution, the secondary market illiquidity as well as the risk-neutral pricing measure are common knowledge. In addition, both shareholders and creditors observe the value process of a firm’s underlying assets and its choice of capital structure.

ASSUMPTION 6: [Private Information] A firm’s type is observable only by its shareholders.

Because all firms have the same rate of growth, \(r - \delta\), and start with the same volatility, \(\sigma_l\), bond investors cannot determine the type of a firm from the value process of its underlying assets alone. Therefore, the asymmetry of information above forces creditors to rely on the firms’ choices of capital structure to infer the riskiness of their underlying projects. However, as I will show, the capital structure choice is not always a sufficient statistic for a firm’s underlying type.

B. Misrepresentation

Notwithstanding their limited oversight, debt holders influence firms’ choices by precluding certain arbitrary deviations. Knowledge of the public distribution of types allows creditors to refuse funding to firms whose capital structure is inconsistent with the firm-value maximization choice for either type. As I will further explain in the next sections, assumptions 1 to 6 imply that creditors can restrict firms’ deviations to the set of policies that render one firm-type indistinguishable from another. In other words, the only way a firm can choose a sub-optimal capital structure is if it misrepresents itself by copying the capital structure of another type.

Misrepresentation can benefit shareholders through its effect over the firm’s debt rollover costs. By leading creditors to believe the volatility of the underlying assets is lower than what it actually is, a misrepresenting firm raises the valuation of its newly issued bonds. Reduced debt rollover costs in turn increase the total value of the firm and the equity valuation with it. While it is true that

\(^{17}\)Numerical results for the optimal risky-type’s payoffs can be found in section B of the Online Appendix for multiple \(\lambda, \sigma_h\) and \(\kappa\) combinations.
the distortion in the mix of equity and debt leads the firm to default sooner, shareholders can gain from the increase in the leverage of the firm beyond what creditors would deem appropriate, as well as the mispricing of the firm’s outstanding debt and newly-issued bonds.

To illustrate shareholders’ incentive to misrepresent their firm’s type, I compute the market-to-book ratio a risky firm would obtain if it managed to deceive creditors by immitating the safe type. In computing the bankruptcy barrier and the payoffs of a risky firm that misrepresents its type, I set its capital structure to the capital structure of the safe type, and let creditors price its newly-issued bonds as if they were issued by a safe type firm.\footnote{A description of the numerical approach to the computation of the equity value in the case of misrepresentation can be found in Online Appendix I.}

Figure 3 in Appendix B.1 plots the optimal firm value and MBR payoffs for the safe (green) and risky types (blue) and contrasts them to the misrepresentation payoffs (red). In this example, the volatility shock intensity is fixed at $\lambda = 0.1$ for all risky firms, so risky types differ only in their post-shock volatility, $\sigma_h$. Since the safe type is not exposed to the shock ($\lambda = 0$), its optimal firm value and MBR are invariant to the shock size (horizontal green lines.) Misrepresentation yields a higher MBR for all risky types (red curve above the blue curve in the bottom graph.) Therefore, if creditors were unable to directly observe a firm’s type, risky firms would misrepresent themselves. Similarly, figure 4 contrasts the optimal firm value and MBR to their corresponding misrepresentation payoffs. This time, however, the risky types’ post-shock volatility is fixed at $\sigma_h = 0.225$, while the shock intensity is allowed to vary. The results are qualitatively the same: misrepresentation increases the risky-type shareholders’ payoffs.

IV. Standardized Bonds in Electronic Platforms

So far, the model has considered only covenant-free bonds trading in one type of secondary market. I now relax the restriction on the types of bonds that a firm can issue and formalize the distinction between competing secondary bond markets. Besides covenant-free bonds, firms can issue bonds with a debt-protective covenant. This covenant is a bond clause that allows debt holders to observe a firm’s type. Although immaterial in a full information setting, I show that this clause commands a premium when bond investors cannot observe a firm’s type.

I consider two types of trading venues, over-the-counter (OTC) markets and electronic trading platforms (EP). They differ from one another in two important ways. First, EPs are deemed more liquid than OTC markets. This is captured in the model by a lower transaction cost parameter $\kappa$, so that $\kappa_{EP} < \kappa_{OTC}$. Secondly, trades in EPs are restricted to a set of standardized bonds. These are covenant-free bonds whose contracts belong to a pre-determined set of contracts $\{b_{EP}^j\}_{j=1}^n$. In contrast, OTC markets accept any kind of bonds, including those with the debt protective covenant. Therefore, despite being more liquid, EPs take away firms’ discretion over the coupon rate (or the $c/p$ ratio) and their ability to tailor debt to allow for greater creditor oversight.

In this setting, standardized debt is traded exclusively in electronic markets. All else constant, the higher liquidity of EPs makes them a more suitable venue for secondary trades in the eyes of the firms. The smaller transaction costs allow for a less expensive portfolio rebalancing, which in turn raises the price investors are willing to pay for newly-issued bonds. Put differently, the more liquid a secondary market is, the lower the firms’ debt rollover costs.\footnote{True enough, the issuance of standardized debt instruments reduces the ability of firms to maximize their total economic value by fine-tuning their leverage. However, changes in the value of the coupon and principal that preserve a bond’s coupon ratio, $c/p$, can be accommodated by an inversely proportional adjustment to the measure of outstanding bonds, $\mu_b$ (see Lemma 1 in Appendix H.) Therefore, only in the extreme case in which the exchange-traded bond’s coupon ratio much differs from that which a firm would optimally choose would the issuance of non-standardized bonds affect the shareholders’ payoffs.}

\begin{figure*}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Figure 3 plots the optimal firm value and MBR payoffs for the safe (green) and risky types (blue) and contrasts them to the misrepresentation payoffs (red). In this example, the volatility shock intensity is fixed at $\lambda = 0.1$ for all risky firms, so risky types differ only in their post-shock volatility, $\sigma_h$. Since the safe type is not exposed to the shock ($\lambda = 0$), its optimal firm value and MBR are invariant to the shock size (horizontal green lines.) Misrepresentation yields a higher MBR for all risky types (red curve above the blue curve in the bottom graph.) Therefore, if creditors were unable to directly observe a firm’s type, risky firms would misrepresent themselves. Similarly, figure 4 contrasts the optimal firm value and MBR to their corresponding misrepresentation payoffs. This time, however, the risky types’ post-shock volatility is fixed at $\sigma_h = 0.225$, while the shock intensity is allowed to vary. The results are qualitatively the same: misrepresentation increases the risky-type shareholders’ payoffs.}
\end{figure*}
Finally, to keep the model tractable, I assume there is a single standardized bond contract, \( b^{EP} \), set to resemble the optimal bond contract that a safe type firm would optimally issue under full information if it were free to choose its debt maturity profile and principal-coupon ratio.

**ASSUMPTION 7:** [Standardized Bonds] There is a single standardized bond contract, \( b^{EP} \), which is derived from the safe-type’s unconstrained optimal bond contract under full information. The \( b^{EP} \) parameters are set as follows:

1. maturity \( m^{EP} \) equals the maturity of the unconstrained bond contract;
2. coupon \( c^{EP} \) is set to the nearest half integer of the unconstrained bond contract coupon;
3. principal \( p^{EP} \) is set to \( c^{EP} \times r^{p} \), where \( r^{p} \) is the nearest half integer of the unconstrained bond contract principal-coupon ratio.

**V. Equilibria in Electronic Platforms**

Since the capital structure is set up front (time 0) and fixed until bankruptcy, firms play a static game. When issuing bonds of a standardized contract, \( b^{EP} \), their only variable of choice is the measure of outstanding bonds, \( \mu_{b} \). As will be demonstrated, the creditors’ oversight, however limited in the presence of asymmetric information, limits firms’ strategies to the pure-strategies defined below.

**DEFINITION 2:** (Type-i’s Pure Strategy in EP) Suppose there is only one secondary bond market, an electronic platform whose uniquely accepted bond contract is \( b^{EP} \). Let the pair of firm types be denoted by \( \gamma \equiv (\gamma_{s}, \gamma_{r}) \), where \( \gamma_{s} \equiv (0, \sigma_{l}) \) and \( \gamma_{r} \equiv (0, \sigma_{h}) \). A strategy for type \( i \), \( i \in \{s, r\} \), is the function \( s_{i} : \mathbb{R}^{+} \to \mathbb{R}^{+} \) that takes type-\( j \)’s measure of outstanding bonds and returns a measure of outstanding bonds \( \mu_{b_{i}} \), like so:

\[
s_{i}(\mu_{b_{j}}; b^{EP}, \gamma, \mu_{s}) = \mu_{b_{i}}
\]

I refer to strategies that maximize the total economic value of the firms as *firm-value-maximizing* strategies:

**DEFINITION 3:** (Firm-Value-Maximizing Strategy) Let the space of feasible strategies to type-\( i \) be \( S_{i} \). A strategy \( s^{*}_{i} : S_{i} \to \mathbb{R}^{+} \) is firm-value-maximizing for type-\( i \) when type-\( j \) plays \( s^{*}_{j} \) if, and only if,

\[
s^{*}_{i} = \arg \max_{s \in S_{i}} \left\{ D(V_{0}, s, s_{j} | \gamma = \gamma_{i}; b^{EP}, \gamma, \mu_{s}) + E \left( V_{0}, s, s_{j} | \gamma = \gamma_{i}; b^{EP}, \gamma, \mu_{s} \right) \right\}
\]

If types are not directly observable, bond investors must infer them from (i) their knowledge of the firm-type distribution, \((\gamma, \mu_{s})\), and (ii) \( \mu_{b} \), the set of observable choices of leverage \( \mu_{b} \) in equilibrium (assumption 5.) I denote by \( p_{i}(\mu_{b}) \) the probability assigned by bond investors that the firm is of type \( i \), conditional on the firm’s choice of bond issuance \( \mu_{b} \).

\[
p_{i}(\mu_{b}) \equiv \text{Prob} (\gamma = \gamma_{i} | \mu_{b})
\]

In the numerical examples that follow, the standardized bond maturity is set to 1 year, the EP transaction costs parameter, \( \kappa^{EP} \), is fixed at 25 basis points, and all other parameters are as in Table I in Appendix A.
A. Full Information Equilibrium

When types are publicly observable, the firm-value-maximizing strategies are independent of the firm type distribution. By assumption 4, firms must set the measure of outstanding bonds to the value of $\mu_b$ that maximizes their total economic value, conditioned on their type, safe or risky. Equation 16 then becomes

$$\mu_b^* (b^{EP}; \gamma_i) \in \arg \max_{\mu_b \in \mathbb{R}_+} \left\{ D \left( V_0; V^B \left( \mu_b, b^{EP}, \gamma_i \right), \mu_b, b^{EP}, \gamma_i \right) + E \left( V_0; V^B \left( \mu_b, b^{EP}, \gamma_i \right), \mu_b, b^{EP}, \gamma_i \right) \right\}. \quad (17)$$

For simplicity, I denote by $MBR^{FI}(\gamma_i)$ the corresponding payoff in a full-information equilibrium:

$$MBR^{FI}(\gamma_i) \equiv MBR \left( V_0; \mu_b^* (b^{EP}; \gamma_i), b^{EP}, \gamma_i \right)$$

Firms then enter the market if, and only if, shareholders’ rate of return at time 0 is strictly positive, that is, $MBR^{FI}(\gamma_i) > 0$. Figures 5 and 6 in Appendix B.2 show the iso-curves for the optimal firm values and market-to-book ratios that would hold in a full information equilibrium for varying risky-types. As expected, the optimal firm value and MBR are strictly decreasing in the volatility shock size, $\sigma_h$, and intensity, $\lambda$.

B. Firms’ Choices under Asymmetric Information

I now turn to the case of asymmetric information between creditors and equity holders. By assumptions 5 and 6, bond investors know the distribution of firms and the market characteristics, but, unlike shareholders, they do not directly observe a firm’s type. Under such conditions, the first-best (Full Information) equilibrium may not be attainable. As discussed in section III, equity investors may benefit from the misrepresentation of their firms’ riskiness. Therefore, to pin down the equilibrium in the electronic market, I start by comparing the payoffs in the full-information equilibrium (FI) to the corresponding payoffs in case of misrepresentation (MP).

Let $MBR \left( V_t; \mu_b|\gamma = \gamma_i; b^{EP}, \mu_s, \gamma \right)$ denote the market-to-book ratio of equity as a function of the value of underlying assets and the measure of outstanding bonds when the firm type is $\gamma_i$, given the standardized bond contract $b^{EP}$ and the distribution of firm types $(\mu_s, \gamma)$. Type-$i$’s misrepresentation market-to-book ratio of equity is:

$$MBR^{MP} (i \rightarrow j) \equiv MBR \left( V_t; \mu_{b,j}^*|\gamma = \gamma_i, \mu_s = 1 - 1_{\{i=s\}}; b^{EP}, \gamma \right)$$

where $1_{\{i=s\}}$ is the indicator function that equals 1 when type $i$ is the safe type. The formula for the measure of safe types in the expression above, $\mu_s = 1 - 1_{\{i=s\}}$, captures the fact that misrepresentation payoffs are computed by assigning probability 1 to the event that a firm that chooses $\mu_{b,j}^*$ is indeed of type-$j$.

Figure 7 in Appendix B.2 shows the iso-curves for the market-to-book ratios of equity in case the risky-type were to copy the capital structure of a safe type. Figures 8 and 9 then show the absolute and percentage differences between these misrepresentation payoffs and the MBR in a full information equilibrium, respectively. The MBR associated with misrepresentation decreases in the volatility shock size, $\sigma_h$, and intensity, $\lambda$. This is to be expected, since the safe type’s (FI) optimal MBR is higher than the corresponding (FI) optimal MBR for risky types. More interestingly, the gain from misrepresentation as a function of the risky-type’s shock exposure is
non-monotonic: for very small shock size and shock intensity, misrepresentation yields a lower MBR gain than for moderate shock exposure. As the shock exposure continues to increase, the gains from misrepresentation turn into losses. Shareholders of firms with higher values of shock intensity and shock size would be worse off from misrepresentation.

B.1. Creditors’ Limited Oversight and Deviations from the Firm-Value-Maximizing Strategy

Despite their inability to distinguish firms’ exposure to the volatility risk, creditors can still preclude arbitrary deviations from the firm-value-maximizing strategy. First, by assumption 5, bond investors know each types’ characteristics and what their corresponding capital structures should be in a full-information equilibrium. In addition, investors do observe the prevailing capital structures in the market and are able to compare a firm’s particular choice of leverage, whatever its type may be, to the that of other firms. In the presence of asymmetric information, assumption 4 becomes

ASSUMPTION 8: [Creditors’ Limited Oversight Condition] Creditors refuse to fund a firm if they can determine its strategy is not firm-value-maximizing.

This assumption greatly constrains the firms’ choices of $\mu_b$. Measures that are inconsistent with the firm-value maximizing capital structures for any type in at least one possible market equilibrium are immediately ruled out.

CLAIM 1: A strategy $s$ that is not firm-value maximizing for either type cannot hold in equilibrium.

Proof. By assumption 8, firms that play a strategy that is incompatible with firm value maximization for either type do not get funded. 

It follows that, in any pooling equilibrium, the equilibrium measure of outstanding bonds $\mu_b^*$ must be firm-value maximizing for at least one type.

COROLLARY 1: In a pooling market outcome, the equilibrium strategy $s^*$ must be firm-value maximizing for at least one firm type.

And that, in a separating equilibrium, both types’ strategies must be firm-value-maximizing.

COROLLARY 2: In a separating equilibrium, both type-contingent strategies $s_i^*$, $i \in \{s,r\}$, are firm-value maximizing for their respective types.

B.2. First-Best Equilibrium under Asymmetric Information

The following proposition establishes that a first-best equilibrium holds under asymmetric information if, and only if, the full-information equilibrium strategies are such that neither type can increase its MBR of equity by misrepresenting itself.

PROPOSITION 1: The full information equilibrium holds under asymmetric information if, and only if,

$$MBR^{FI}(\gamma_i) \geq MBR^{MP}(i \rightarrow j) \text{ for } i, j \in \{s, r\} \text{ and } i \neq j$$

Proof. Let $(\mu_{b,s}^*, \mu_{b,r}^*)$ be the full-information equilibrium pair of measures of outstanding bonds, where subscripts $s$ and $r$ stand for safe and risky, respectively. Suppose $s_i^* = \mu_{b,i}^*, i \in \{s, r\}$, is a (separating) equilibrium strategy under asymmetric information and consider a deviation $s_j^*=
\( \mu_b \notin \{ \mu^*_b,s, \mu^*_b,r \} \), for some firm of type \( j \). By the Creditor’s Limited Oversight Funding Condition (assumption 8), this firm will not be funded, regardless of its type. Therefore,

\[
MBR (\mu_b | \gamma = \gamma_i; b^{EP}, \mu_s, \gamma) = 0 \text{ if } \mu_b \notin \{ \mu^*_b,s, \mu^*_b,r \}, \text{ for } i = s, r
\]

Because \( MBR (\mu_b | \gamma = \gamma_i; b^{EP}, \mu_s, \gamma) \leq MBR^{FI} (\gamma_i) \), the truth-telling strategy is weakly preferred to any such deviation. It follows that the only possible deviation for the type \( j \) is to choose type-\( i \)'s measure of outstanding bonds, \( \mu^*_b,i \), which by assumption also yields a weakly lower return to its shareholders. Since no deviation provides a higher payoff to shareholders, the truth-telling strategies are an equilibrium.

Conversely, suppose \( s^*_i = \mu^*_b,i, i \in \{ s, r \} \), is a (separating) equilibrium strategy under asymmetric information but there exists a misrepresenting strategy \( s_j \) for some type \( j \) such that

\[
MBR^{MP} (j \rightarrow i) > MBR^{FI} (\gamma_j), \text{ for } i, j \in \{ s, r \} \text{ and } i \neq j
\]

The best response of type-\( j \)'s shareholders to type-\( i \)'s truth-telling strategy is to play \( s_j = \mu^*_b,i \), so the full-information equilibrium cannot hold. Contradiction!

C. Pooling v.s. Separating Market Outcomes

In this section, I begin the analysis of the equilibrium properties in Electronic Platforms by studying the best responses of the safe-type firms to a misrepresentation strategy by the risky firms, conditional on a particular market outcome. A market outcome refers to the differentiability of firm types under asymmetric information. I call it a pooling market outcome any event event in which the safe and risky types choose the same measure of outstanding bonds, that is, \( s_s = s_r \). Conversely, I call it a separating market outcome any event in which these measures differ.

Risky firms will attempt to misrepresent themselves as safe if, by doing so, they can increase their shareholders’ rate of return. Since the types distribution is common knowledge, creditors revise their valuation of all outstanding debt accordingly. The lower prices of newly-issued bonds in turn raise the debt rollover costs of the safe type firms, thereby decreasing their equity and initial firm valuations. In anticipation of this, safe-type shareholders lower their firm’s leverage as they enter the market.

Safe-type firms can either accommodate the move by the risky firms and adjust their leverage just enough so that, in equilibrium, types are indistinguishable, or they can change their measure of outstanding bonds to discourage misrepresentation all together. In the first case, safe and risky firms are pooled together and creditors price newly-issued debt by averaging their valuations of safe and risky bonds (see section V.D). The second market outcome is a separating arrangement in which the safe type’s capital structure is distorted to ensure misrepresentation yields a lower market-to-book ratio to risky firms. Whether a pooling or separating equilibrium prevails depends on the safe-type firms’ initial valuation under each market arrangement.

To determine the equilibrium, I compute the safe-type’s best responses conditional on a particular type of market outcome. By claim 1 and its corollaries, these are the strategies that maximize the safe-type’s firm value conditional on risky firms choosing the same capital structure, in case of pooling, or on risky firms maximizing their initial valuation, in case of separating arrangement.

Definition 4 (Best conditional responses): The safe type’s best conditional response to the risky type’s misrepresentation is the strategy \( s^*_c \) that maximizes the safe type’s firm value conditional on a particular market outcome, either pooling or separating.
Figure 10 in Appendix B.2 plots the safe type’s initial firm valuation if safe firms were to play the best pooling-conditional strategy. As before, the safe type is fixed ($\sigma_l = 0.15$), and results are computed for multiple risky types, varying in their exposure to the volatility shock size, $\sigma_h$, and intensity, $\lambda$. Conditional responses assume only one risky type at a time. The measure of entrant safe-type firms is set to $\mu_s = 0.2$, and all remaining parameters are as in Table I in Appendix A. By definition 4 above, the best pooling-conditional response is the measure of outstanding bonds $\mu_b$ that solves

$$\max_{s \in \mathbb{R}^+} \left\{ D \left( V_0, s, s; V^B, \gamma, \mu_s | \gamma = \gamma_s \right) + E \left( V_0, s, s; V^B, \gamma, \mu_s | \gamma = \gamma_s \right) \right\}$$

Figures 11 and 12 show the absolute and percentage firm value differential relative to the Full Information equilibrium, respectively. Not surprisingly, the safe type’s maximal firm value in a pooling market outcome is strictly decreasing in the risky type’s volatility shock exposure. As the initial safe-type firm valuation decreases, so does its rate of return to safe type’s shareholders (figures 13 to 14.)

Turning now to the payoff of risky firms, figures 15 to 17 plot the MBR of risky-type firms in a pooling outcome and compare them to their corresponding payoffs in a full-information equilibrium. The MBR in a pooling outcome is higher the smaller the risky type’s overall exposure to the volatility shock. However, the gain in MBR is proportionally higher for firms with moderate exposure (figure 17.) Evidently, risky firms can always choose the truth-telling, firm-value-maximizing strategy enforced in a full information equilibrium, $s^{FI}_r$. Therefore, a pooling market outcome can only hold if the MBR differential is strictly positive:

$$MBR^{POOL}_r > MBR^{FI}_r$$

If a separating equilibrium is to hold, safe types must adjust their capital structure to ensure risky firms cannot increase their MBR by means of misrepresentation. In this case, the best separating outcome conditional response is the solution to

$$\max_{s \in \mathbb{R}^+} \left\{ D \left( V_0, s, s^{FI}_r; V^B, \gamma, \mu_s | \gamma = \gamma_s \right) + E \left( V_0, s, s^{FI}_r; V^B, \gamma, \mu_s | \gamma = \gamma_s \right) \right\}$$

s.t.

$$MBR \left( s, s | \gamma = \gamma_r \right) \leq MBR \left( s, s^{FI}_r | \gamma = \gamma_r \right) \quad \text{(IC)}$$

The (IC) restriction above is the incentive-compatibility constraint that ensures risky type’s shareholders won’t benefit from their firms copying the capital structure of the safe firms. Notice that, by corollary 2 to claim 1, a separating outcome requires that risky firms play their full information firm-value-maximizing strategy, $s^{FI}_r$: since creditors can differentiate types, a risky firm that chooses any other capital structure does not get funded.

Similarly to the pooling outcome payoff plots, figures 18 to 20 show the initial firm valuation and the absolute and percentage firm value differentials relative to a full information equilibrium when safe type firms play the best separating-outcome conditional response. The lower the exposure of the risky type to the volatility shock, the higher must be the distortion to the safe type’s capital structure to ensure separation. Conversely, when the risky firms’ exposure to the shock is sufficiently high, misrepresentation yields a lower MBR to risky firm shareholders. In this case, no adjustment is necessary and the full-information equilibrium hold. (Compare the areas in figures 19 and 20 for which the differentials are null to the corresponding areas in the misrepresentation MBR differentials plots in figures 8 and 9.)

Figures 21 to 22 show the impact of the incentive-compatibility condition over the safe type’s shareholders’ rate of return. The MBR is proportionately more affected when the risky type has
moderate exposure to the volatility shock. Finally, since risky types play the full information firm-value-maximizing strategy, \( s^{FI}_r \), their payoffs always coincide with those in a full information equilibrium (figures 23 to 25).

D. Valuation in a Pooling Market Outcome

In a pooling market outcome, creditors are initially unable to determine the type of any given firm in an entrant cohort. I refer to these firms whose types are not yet known by bond investors as undisclosed-type firms. Over time, the measure of firms in this category shrinks, as more and more firms have their types revealed. In this process, the ratio of safe-to-risky firms among the undisclosed-type firms in the same cohort changes. This poses a problem to the analysis, as it leads to type-specific bankruptcy barriers that vary with the age of the cohort their firms’ belong to. In this section, I explain how pooling affects bond valuation and discuss how I simplify the analysis to keep the model tractable.

There are two ways in which types are disclosed over time. The first is through the arrival of a volatility shock, since shocks are public information (see assumption 5.) A risky firm hit by a shock experiences a permanent increase in its volatility, and creditors adjust its debt valuation accordingly. The second way happens when the firm’s fundamental value, \( V_t \), crosses the highest of the type-specific default barriers from above for the first time. More formally, denote by \( a \geq 0 \) the age of a firm cohort at time \( t \), and let \( V^B_i(a|\mu_b, b^{EP}, \gamma, \mu_s) \) be the default barrier for a type-\( i \) firm of age \( a \), for \( i \in \{s, r\} \). Define \( V^B(a|\mu_b, b^{EP}, \gamma, \mu_s) \) as maximum of the two barriers:

\[
V^B(a|\mu_b, b^{EP}, \gamma, \mu_s) \equiv \max \{ V^B_s(a|\mu_b, b^{EP}, \gamma, \mu_s), V^B_r(a|\mu_b, b^{EP}, \gamma, \mu_s) \}
\]

Now suppose \( V^B(a|\mu_b, b^{EP}, \gamma, \mu_s) = V^B_r(a|\mu_b, b^{EP}, \gamma, \mu_s) \), and consider the case of an undisclosed-type firm whose fundamental value hits this threshold as the firm reaches age \( a \). If the firm is not declared bankrupt, creditors’ know its is of the safe type. From this point on, the uncertainty about this particular firm’s risk exposure and bankruptcy policy vanishes.

Because both safe and risky firms start with low volatility, all undisclosed-type firms face the same probability of hitting the upper default barrier. Consequently, even though the crossing of this barrier reveals private information about a firm, it does not affect the ratio of safe-to-risky firms among the remaining undisclosed-type firms in the same cohort. Only the arrival of a volatility shock alters this ratio, since it affects exclusively one of the types. Whenever a risky firms suffers the shock, creditors must revise their expectations about the type-distribution of the remaining undisclosed-type firms in the same cohort. In addition, since the shocks follow a Poisson process, the safe-to-risky ratio obeys a deterministic process that depends only on the initial measure of safe firms, \( \mu_s \), and the age of the cohort.

**Proposition 2:** The probability that an undisclosed-type firm in a cohort of age \( a \geq 0 \) is of the safe type is:

\[
p_s(a) = \frac{\mu_s}{\mu_s + (1 - \mu_s) \cdot e^{-\lambda a}}
\]

**Proof.** See Appendix J.1. \( \square \)

Notice that, as time goes by, the likelihood of an undisclosed-type firm being of the safe type goes to 1, as expected:

\[
\lim_{a \to \infty} p_s(a) = 1
\]
The bond price of a undisclosed-type firm in a pooling market outcome can be computed as the sum of the type-contingent bond prices, weighted by the probability of each type in the cohort:

$$d \left( V_t; \tau; a_t, \mu_b, b^{EP}, \gamma, \mu_s \right) = p_s \left( a_t \right) d \left( V, \tau; V_s^B \left( a_t | \mu_b, b^{EP}, \gamma, \mu_s \right), \mu_b, b^{EP}, \gamma_s \right) + (1 - p_s \left( a_t \right)) d \left( V_t, \tau; V_r^B \left( a_t | \mu_b, b^{EP}, \gamma, \mu_s \right), \mu_b, b^{EP}, \gamma_r \right)$$ (20)

The time-varying, type-specific bankruptcy barriers greatly complicate the numerical derivations of the optimal debt and equity prices. To avoid this problem, I simplify the analysis by setting $p_s \left( a_t \right)$ to $\mu_s$, for all $a \geq 0$. By making this probability independent of the age of the cohort, I once more obtain time-invariant bankruptcy barriers. This in turn allows me to derive equity prices through a simple adaptation of the numerical method used in the computation of the risky-type’s equity price under full information (see Appendix J.) It must be noted, however, that this simplification lowers the value of bonds, because the weight assigned to the more valuable, safe-type's bond price in the formula above is kept fixed at $\mu_s$, which is strictly lower than $p_s \left( a_t \right)$ for all $a > 0$ (Proposition 2.) Despite leading to an overestimation of the safe firms’ debt rollover costs in any pooling market outcome, this assumption does not alter the model’s qualitative results.

E. Pinning down the Equilibrium

Having determined the only two possible market outcomes, I now turn to the computation of the prevailing equilibrium in an electronic platform. The algorithm to determine the equilibrium under asymmetric information starts by backing out the full-information, firm-value-maximizing strategies, $(s^F_t, s^F_r)$. If neither firm type can increase its market-to-book ratio of equity by copying the capital structure of another, then a first-best equilibrium prevails. Any deviation from the type-contingent optimal measures of outstanding bonds not only reduces the rate of return to shareholders, but is de facto ruled out by the creditor’s funding condition. When, however, misrepresentation yields a higher MBR, the first-best equilibrium is no longer attainable. As shown numerically, this is the case for risky firms with small to moderate exposure to the volatility shock. Since the distribution of types is common knowledge, creditors anticipate this misrepresentation and revise their valuation of newly-issued bonds. Faced with the prospect of increased debt rollover costs, safe firms in turn adjust their leverage.

Conditional on a pooling market outcome, safe firms’ best response to a misrepresentation strategy by the risky type is to choose a measure of outstanding bonds that maximizes their initial firm valuation when bonds prices are an weighted average of type-specific (full information) bond values (definition 4.) In a separating market outcome, the safe type’s best response maximizes its firm value while discouraging misrepresentation.

The difference between the safe firm’s initial valuations in the full and asymmetric information equilibria is the informational cost of adverse selection in bond markets:

$$INFC \left( \mu_s, \gamma; \kappa^{EP}, b^{EP} \right) = FV^F \left( s^{FI}_s | \mu_b, \kappa^{EP}, b^{EP}, \gamma = \gamma_s \right) - FV \left( s^*, s| \mu_s, \gamma, \kappa^{EP}, b^{EP}, \gamma = \gamma_s \right)$$ (21)

where $s^{FI}_s = \mu_b \left( b^{EP}, \gamma_s \right)$ is the safe type’s optimal measure of outstanding bonds in a full-information setting, as defined in equation 17. I assume safe types choose the market outcome that minimizes the informational cost. In other words, whenever risky firms attempt to misrepresent themselves, creditors enforce the strategy that maximizes safe firms’ firm value out of the two best contingent responses.20 By claim 1, the demand for debt issued by firms that choose a capital structure other than the equilibrium ones is zero.

20Alternatively, one could assume creditors enforce the market outcome that maximizes the average firm value.
Figures 26 and 27 show the safe type’s initial firm valuation and the risky type’s market-to-book -ratio of equity in equilibrium, respectively, when the only secondary market is the electronic platform (EP.) The light gray area on the LHS plots indicate the set of risky types for which a pooling equilibrium would prevail. When the risky type’s exposure to the volatility shock is small to moderate, safe type’s can maximize their initial firm valuation by accommodating the risky firms’ misrepresentation. As the volatility risk increases, (i) the impact of a pooling market outcome on the safe type’s debt rollover costs is raised, while (ii) the adjustment in the safe type’s leverage necessary to discourage misrepresentation by risky firms is falls. Safe firms then maximize their initial firm valuation by increasing their leverage just enough to leave shareholders of risky firms indifferent between playing the truth-telling, firm-value maximizing strategy, \( s_r^{FI} \), or misrepresenting their firms’ type (medium gray and dark areas.) The dark areas indicate the set of risky types with such a high exposure to the shock that a misrepresentation strategy returns a lower MBR. No adjustment from the safe firms is necessary and a first-best, full information equilibrium prevails, even though types are not directly observable by creditors.

VI. Endogenous Covenants and Over-the-Counter Markets

I now return to the discussion of competing trading venues. Despite offering higher liquidity, electronic exchanges do not necessarily preempt trade in over-the-counter markets. In this section, I discuss the implications of debt standardization for firms’ ability to signal their creditworthiness.

I assume firms can fully reveal their type to bond investors by means of a debt protective covenant. This covenant is a clause that is added to a bond contract, \( b \), which allows creditors to observe a firm’s exposure to the volatility shock, thereby shielding them from type-misrepresentation strategies. Due to the restrictions over the types of securities transacted in EPs, this signaling mechanism comes only at a price. On the one hand, the debt protective covenant eliminates the informational costs in secondary markets. On the other, it raises firms’ debt rollover costs by constraining secondary trades to OTC markets. When issuing debt, therefore, firms must balance out the benefits of signaling their riskiness with the liquidity gains offered by electronic exchanges.

I call dual-market valuation differential the difference between the safe type’s initial firm valuation in an EP-only equilibrium (as defined in the previous section) and its corresponding value in an OTC-only equilibrium:

\[
\Delta FV(\mu_s, \gamma, \kappa, b^{EP}) \equiv FV(s_s^*, s_r^*, \mu_s, \gamma, \kappa^{EP}, b^{EP}, \gamma = \gamma_s) - FV(s_s^{OTC,*}, \kappa^{OTC}, \gamma = \gamma_s)
\]  

where \( \kappa \equiv (\kappa^{EP}, \kappa^{OTC}) \), and \( s_s^{OTC,*} \) is the safe-type’s equilibrium strategy in OTC markets. Notice that, because bonds trading in OTC markets can carry the debt protective covenant at no additional cost, any equilibrium in these secondary trading venues is a full-information equilibrium. Put differently, the informational cost of adverse selection is zero for over-the-counter trades.

To study the determinants of equilibria across competing secondary markets, I decompose the valuation differential into a liquidity and an informational components. The liquidity term is defined as the gain in the safe type’s initial firm valuation in a full-information equilibrium when secondary trades transition from over-the-counter markets to electronic exchanges:

\[
LQD(\gamma_s, \kappa, b^{EP}) \equiv FV^{FI}(s_s^{FI,*}, \kappa^{EP}, b^{EP}, \gamma = \gamma_s) - FV(s_s^{OTC,*}, \kappa^{OTC}, \gamma = \gamma_s)
\]  

In this case, a separating equilibrium would hold whenever

\[
\mu_s \times FV_r(s_r^{SEP}) + (1 - \mu_s) \times FV_r(s_r^{SEP}) \geq \mu_s \times FV_r(s_r^{POOL}) + (1 - \mu_s) \times FV_r(s_r^{POOL})
\]

where \( s_r^{SEP} \) and \( s_r^{POOL} \) are the safe type’s best conditional responses (definition 4) in a separating and pooling market outcomes, respectively.
This component captures the effect of the transaction costs parameter $\kappa$ in the total return of an investment in a safe project, absent any asymmetry of information between creditors and equity investors. Because it focuses on valuation in full information settings, the liquidity term is independent from the measure of safe firms, $\mu_s$.

By the definition of the information cost of adverse selection in equation 21 and the expression above, the cross-market variation in the total returns can be expressed as the difference between the liquidity gains in EP relative OTC markets and the cost of adverse selection associated with unsecured, standardized debt.

$$
\Delta FV (\mu_s, \gamma, \kappa, b^{EP}) = (FV \left( s^*_s, s^*_r | \kappa^{EP}, b^{EP}, \gamma = \gamma_s \right) - FV^{FI} \left( s^F_s, s^F_r | b^{EP}, \gamma = \gamma_s \right))
- (FV \left( s^*_s, s^*_r | \kappa^{OTC}, \gamma = \gamma_s \right) - FV^{FI} \left( s^F_s, s^F_r | b^{EP}, \gamma = \gamma_s \right))
= LQD \left( \gamma_s, \kappa, b^{EP} \right) - INFC \left( \mu_s, \gamma, \kappa^{EP}, b^{EP} \right)
$$

Safe firms maximize their initial firm valuation by issuing standardized bonds whenever this differential is positive. Conversely, if the informational cost in electronic exchanges is greater than the dual-market liquidity differential, creditors require safe firms to issue bonds with a debt protective covenant that fully reveals their type. In this case, a separating, dual-market equilibrium holds, in which only risky firms’ bonds are traded in the exchange.

Figure 28 in Appendix B.2 extends the analysis to allow for trading in the more illiquid over-the-counter markets ($\kappa^{OTC} = 32.5$ b.p.). The iso-curve levels are the same as in figure 26, and are contrasted to the safe type’s optimal firm value in OTC, $FV^{OTC}_s = 111.41$. As before, the colored regions on the LHS plot indicate the set of risky firms for which a particular type of equilibrium would prevail. When the risky firms’ exposure to the volatility shock is small, the effect of misrepresentation over firms’ debt rollover costs is modest enough that types pool together, and secondary trading is limited to standardized debt in electronic exchanges. As the volatility risk increases, however, so do the informational costs in EPs. Once these costs surpass the cross-market liquidity differential, safe firms find it optimal to fully reveal their types by issuing non-standardized bonds. This leads to a separating equilibrium in which debt from safe firms is traded over the counter, while risky firms’ bonds are transacted in electronic platforms. Notice how the levels of the iso-curves in the light gray area (OTC) are below the safe firm’s valuation in OTC. But the effect of the volatility risk exposure on the informational costs is non-monotonic. Once the risky type’s exposure to the volatility shock becomes large enough, the informational costs in EP can be minimized by having safe firms switch to a separating market outcome strategy. From this point on, the EP informational costs function is decreasing in the risky firms’ exposure to the shock. As these costs shrink, eventually the cross-market valuation differential becomes positive again, and an EP-only, separating equilibrium holds. Once informational costs reach zero, no adjustment from the safe firms is necessary and a first-best, full information equilibrium prevails in EP.

A. The Effect of the Measure of Safe Types over the Equilibrium Properties

The cross-market liquidity gains are driven by the market-specific transaction costs parameter values, $\kappa^{EP}$ and $\kappa^{OTC}$, but unaffected by the measure of safe firms or the risky type’s exposure to the volatility shock. Conversely, the informational costs of adverse selection are determined by the firm type distribution, while independent from the cross-market transaction costs differential. The equilibrium analysis so far has studied how informational costs, and therefore the dual-market valuation differential, vary with the size and intensity of volatility shocks, while assuming a constant measure of safe firms ($\mu_s = 0.2$). I now extend the analysis by varying both the overall shock risk and share of risky firms in an entrant cohort.
The effect of the risky type’s overall exposure to the volatility shock over firms’ payoffs can be reduced to a one-dimensional problem. The iso-curves in the plots analyzed in the previous sections show that, by fixing one parameter, either \( \lambda \) or \( \sigma_h \), and varying the other, one obtains any equilibrium outcome. To see this, consider a cross-section of the safe type’s firm value function along, say, \( \lambda = 0.25 \) in figure 28, and notice how the resulting surface intersects the varying iso-curves and market equilibrium regions simply. Therefore, in what follows I fix the shock intensity at \( \lambda = 0.3 \).

Figures 29 to 52 in Appendix B.3 repeat the analysis of the firm value and MBR functions in the previous plots, but this time with \( \mu_s \) in the y-axis. Notice how the iso-curves in the full-information equilibrium plots are vertical, as the types’ firm-value-maximizing strategies are independent from one another. Moreover, since misrepresentation payoffs are obtained by assigning probability 1 to firms being of the safe type, by construction the iso-curves in figures 31 to 33 also are vertical. Furthermore, this independence from \( \mu_s \) is also observed in the safe- and risky-types’ firm value and MBR functions in a separating market outcome (figures 42 to 49). The result holds because the incentive-compatibility constraint in the safe firm’s best separating outcome conditional response is independent of \( \mu_s \). In all of these cases, the iso-curves’ levels match the values of a cross-section of the corresponding surfaces along \( \lambda = 0.3 \) in the plots in Appendix B.2.

In contrast, the safe-type’s payoffs in a pooling market outcome are directly affected by the ratio of safe to risky firms in entrant cohorts. As the share of safe firms falls, the informational costs of adverse selection increase, affecting the safe firms’ initial valuation and MBR (figures 34 to 38.) The lower \( \mu_s \) (and the higher the exposure of the risky type to the volatility shock), the more creditors revise down their bond valuation, thereby raising the debt rollover costs for safe firms. Moreover, the concave shape of the iso-curves indicates that the sensitivity of the safe-type’s initial firm valuation to \( \mu_s \) is higher when the risky firms’ exposure to the volatility shock is small to moderate. Interestingly, the risky-type shareholders’ rate of return show little sensitivity to changes in the ratio of safe to risky firms in a pooling market outcome. The nearly vertical iso-curves of the MBR function in figures 39 to 41 suggest that the effects of \( \mu_s \) over the risky firms’ debt rollover costs nearly cancel each other out. On the one hand, an increase in \( \mu_s \) benefits risky firms through its positive impact on creditors’ bond valuation. On the other hand, the higher bond values prompt safe firms to adjust their capital structure by issuing more bonds. The increased leverage then raises the expected debt rollover costs to risky firms.

Figures 50 and 51 show the safe type’s initial firm valuation and the risky type’s market-to-book -ratio of equity in equilibrium, respectively, when the only secondary market is the electronic platform (EP.) As before, when the risky firms’ exposure to the volatility shock is small, a pooling market outcome yields a higher initial firm valuation to the safe type. The informational costs of adverse selection in an EP-only equilibrium increase with the risky type’s volatility shock exposure until the losses to the safe firms’ initial value under a pooling market outcome equal those in a separating market arrangement. From this point on, a separating equilibrium prevails and the informational costs decrease with the risk exposure \( (\sigma_h) \), until eventually reaching zero.

The novelty here is the effect of the measure of safe firms over the risk-exposure threshold marking the transition from pooling to separating equilibrium. While informational costs in a separating market outcome are unaffected by \( \mu_s \), an increase in the measure of safe firms lowers the informational costs in a pooling market arrangement. The higher \( \mu_s \), the lower the effect of the risky type’s misrepresentation over bond prices, and thus on the safe type’s debt rollover costs. Consequently, the larger is the set of risky types for which a pooling equilibrium holds (and hence the concave shape of the separating equilibrium region.) By the same logic, the dual-market equilibria

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\[ \text{Recall the distinction between market outcome and market equilibrium in section V.C.} \]
are more likely to hold the smaller is the measure of safe firms. The light gray area on the LHS plot of figure 52 indicates the combinations of risk exposure and $\mu_s$ for which the EP iso-curve levels are below the safe firm’s valuation in OTC, $FV_{sOTC} = 111.41$. In this region, therefore, the safe firms’ best response is to issue non-standardized bonds with the debt protective covenant.

VII. Conclusion

In recent years, electronic trading in corporate bond markets has seen a substantial and sustained growth. Over 90% of small ticket trades are now done in exchanges. Nonetheless, the bulk of the notional volume traded daily is concentrated in larger ticket size trades, which are still done over-the-counter. The main obstacle to the electronic trading of corporate bonds is arguably the fragmentation of the trading activity across a vast universe of securities. In other asset classes, this electrification process has been accompanied by an standardization effort to facilitate pricing and expand the base of potential investors, thus improving liquidity in secondary markets. Because bonds are information-sensitive securities, however, standardization comes at a cost. Doing away with debt protective covenants can hamper firms’ ability to properly signal their creditworthiness, with non-trivial effect on firms’ credit spreads and debt rollover costs in primary markets.

In this paper, I analyzed the interplay between liquidity gains, arising from changes in the micro-structure of secondary markets, and the informational costs of debt standardization, from a theoretical perspective. To do so, I proposed a structural model of credit risk with asymmetric information and competing secondary markets, where debt covenants arise endogenously. The model features two classes of investors, bond investors (or creditors) and equity holders (shareholders), which invest in firms that can be either safe or risky, depending on their exposure to an idiosyncratic, unhedgeable risk. Debt is traded in competing secondary markets, which differ in (i) their (external) liquidity and (ii) the types of bonds they accept. Electronic platforms (EPs) offer lower transaction costs, but intermediate only trades of standardized, covenant-free bonds, whereas over-the-counter (OTC) markets accept any type of bond.

All else constant, debt issued with standardized, covenant-free bonds is more valuable because these bonds are traded in the more liquid electronic markets. The reduced transaction costs of secondary trades in EPs increase the price of newly-issued bonds, thereby lowering firms’ debt rollover costs. When both classes of investors are fully informed about the firms’ risk exposure, therefore, firms issue standardized debt and all secondary trades happen in electronic exchanges. This result, however, may not hold in the presence of information asymmetry in credit markets.

When bond investors are unable to directly observe firms’ risk exposure, shareholders of riskier firms may increase their rate of return by misrepresenting their firms’ creditworthiness. Type-misrepresentation is akin to an asset substitution problem, wherein creditors’ valuation of a firm’s debt is incommensurate with the firm’s riskiness. However, so long as bond investors are knowledgeable of the distribution of firm types, misrepresentation prompts them to revise downwards the valuation of all standardized bonds. This raises the debt rollover costs for safe firms, which in turn adjust their capital structure, either by increasing their measure of outstanding bonds to discourage the risky type’s misrepresentation, or by reducing their leverage to minimize the impact of the misrepresentation over their debt-rollover costs. Alternatively, safe firms may opt for issuing bonds with a debt protective covenant to signal their creditworthiness.

The direction of the safe type’s leverage adjustment and choice of debt instrument in response to the risky type’s misrepresentation depend on (i) the informational costs of adverse selection in electronic markets (INF C), and (ii) the liquidity differential between the competing secondary trading venues (LQD.) The lower the ratio of safe to risky firms or the higher the risky firms’ exposure
to the unhedgeable shock, the more the safe type's debt rollover costs are affected when pooling together with the risky firms. When the unhedgeable risk differential between the two types of firms is small or the ratio of safe-to-risky firms is sufficiently high, the INFC is minimized by having safe firms reduce their leverage so that types pool together. When the risky differential is large or the measure of safe firms is small, however, the effect of pooling over the safe firms' debt rollover costs is so high that these firms find it preferable to increase their leverage to discourage the risky type's misrepresentation. Covenants arise endogenously as a means of mitigating the informational problem. When informational costs exceed the liquidity differential between over-the-counter and electronic markets, safe firms forego the liquidity gains and issue instead non-standardized bonds to signal their creditworthiness. In this case, a dual-market separating equilibrium holds where only risky firms issue standardized bonds.

The results have implications for (i) the volume of trades in corporate bond exchanges and (ii) the composition of debt across competing secondary trading venues. When setting their clearing and trading fees, electronic platforms must consider the trade-off between liquidity and the costs of bondholder-stockholder conflicts. Debt standardization may exacerbate informational problems, partly offsetting the liquidity gains offered by the centralized trading of a reduced number of securities. In the most severe cases, the resulting informational costs may drive safer firms away from the new electronic platforms, leading to a smaller base of potential clients and reduced revenues.

References


## Appendix A. Tables

**Table I**: Calibrated Parameters

<table>
<thead>
<tr>
<th>General Environment</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest Rate ( r )</td>
<td>8.0%</td>
</tr>
<tr>
<td>Debt Tax Benefit Rate ( \pi )</td>
<td>27%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Firm Characteristics</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Volatility ( \sigma_l )</td>
<td>0.15</td>
</tr>
<tr>
<td>Bankruptcy recovery rate ( \alpha )</td>
<td>60%</td>
</tr>
<tr>
<td>Dividend Payout rate ( \delta )</td>
<td>2%</td>
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</table>

<table>
<thead>
<tr>
<th>Bond Market Illiquidity</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Liquidity shock intensity ( \xi )</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Debt Structure</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity ( m )</td>
<td>1</td>
</tr>
<tr>
<td>Current Fundamental ( V_0 )</td>
<td>100</td>
</tr>
</tbody>
</table>

The table shows the parameter values taken from Table I in *He and Xiong (2012)*, calibrated to firms with a speculative-grade BB rating, as well as the value of transaction costs parameters in electronic platforms (in basis points) used in the analysis.
Appendix B. Plots

B.1 Misrepresentation

Figure 3. Volatility Shock Size and The Incentive to Misrepresent the Firm Type

The figures contrast the optimal firm values (top) and market-to-book ratios (bottom) payoffs to their corresponding values in case of misrepresentation. The debt maturity is fixed at 1 year, and the optimal payoffs are those obtained when the capital structure satisfies the creditors’ funding condition, according to definition 1 in section II.E. The misrepresentation strategy consists in copying the capital structure of the safe type (indicated by the circles $S$.) The volatility shock intensity is fixed at $\lambda = 0.1$ for all risky firms, so risky types differ only in their post-shock volatility, $\sigma_h$. Since the safe type is not exposed to the shock ($\lambda = 0$), its optimal firm value and MBR are invariant to the shock size (horizontal green lines.) In this example, misrepresentation yields a higher MBR for all risky types (red curve above the blue curve in the graph at the bottom.)
The figures contrast the optimal firm values (top) and market-to-book ratios (bottom) payoffs to their corresponding values in case of misrepresentation. The debt maturity is fixed at 1 year, and the optimal payoffs are those obtained when the capital structure satisfies the creditors’ funding condition, according to definition 1 in section II.E. The misrepresentation strategy consists in copying the capital structure of the safe type (indicated by the circles $S$.) The post-shock volatility is fixed at $\sigma_h = 0.225$ for all risky firms, so risky types differ only in their shock intensity, $\lambda$. Since the safe type is not exposed to the shock ($\sigma_h = \xi$), its optimal firm value and MBR are invariant to the shock size (horizontal green lines.) As in figure 3, misrepresentation yields a higher MBR for all risky types (red curve above the blue curve in the graph at the bottom.)
B.2 Contour Plots - $\sigma_h \times \lambda$

### B.2.1 Full Information Equilibrium

#### Figure 5. Optimal Firm Values in an Electronic Platform Full Information Equilibrium

The figure above shows the optimal firm values in an Electronic Platform full information equilibrium for varying $(\lambda, \sigma_h)$-type firms. The transaction costs parameter $\kappa$ is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E. As expected, the optimal firm value is strictly decreasing in the volatility shock size and intensity.
The figure above shows the optimal market-to-book ratios (MBR) in an Electronic Platform full information equilibrium for varying \((\lambda, \sigma_h)\)-type firms. The setting is the same as in figure 5. The transaction costs parameter \(\kappa\) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \((p/c)\) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E. As expected, the optimal MBR is strictly decreasing in the volatility shock size and intensity.

Obs: Recall that the optimal payoffs are those that correspond to the firm-value maximizing capital structure, as defined in section ref:xxx. Therefore, the optimal ratios above correspond to the risky types’ MBR when their firm values are as in figure 5.
B.2.2 Misrepresentation

Figure 7. Misrepresentation Market-to-Book Ratios in an Electronic Platform

The figure shows the market-to-book ratios (MBR) in an Electronic Platform for varying \((\lambda, \sigma_h)\)-type firms if these firms misrepresented themselves by copying the full-information capital structure of the safe type.

The setting in the figure above is the same as in figure 5. The transaction costs parameter \(\kappa\) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \((p/c)\) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

Risky types will choose to misrepresent themselves if (i) creditors cannot observe their types, and (ii) the MBR yielded by copying the capital structure of the safe type exceeds their optimal MBR.
Figure 8. Risky Type’s MBR Differential in EP - Misrepresentation v.s. Full Information Equilibrium Payoffs

The figure shows the difference in market-to-book ratios (MBR) yielded by a type-misrepresentation and a truth-telling (Full Information) strategies in an Electronic Platform, for varying \((\lambda, \sigma_h)\)-type, that is,

\[
MBR^\text{MP}_r - MBR^\text{FI}_r
\]

where subscript \(r\) stands for “risky”, and superscripts \(\text{MP}\) and \(\text{FI}\) stand for “misrepresentation” and “full information”, respectively. The type-misrepresentation strategy consists in copying the full-information capital structure of the safe type.

The setting is the same as in figure 5. The transaction costs parameter \(\kappa\) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \((p/c)\) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

Risky types will choose to misrepresent themselves if (i) creditors cannot observe their types, and (ii) the MBR differential above is strictly positive. The iso-curves above show that the misrepresentation differential is proportionately higher for small shock size or for small shock intensity. Notice however that the impact of the shock exposure is not monotonic: for very small shock size and shock intensity, misrepresentation yields a lower MBR (orange-colored, triangular region in the southwest corner.)
Figure 9. Risky Type’s MBR Percentage Differential in EP - Misrepresentation v.s. Full Information Equilibrium Payoffs

The figure shows the percentage difference in market-to-book ratios (MBR) yielded by a type-misrepresentation and a truth-telling (Full Information) strategies in an Electronic Platform, for varying \((\lambda, \sigma_h)\)-types, that is,

\[
\frac{MBR_{MP} - MBR_{FI}}{MBR_{FI}}
\]

where subscript \(r\) stands for “risky”, and superscripts \(MP\) and \(FI\) stand to “misrepresentation” and “full information”, respectively. The type-misrepresentation strategy consists in copying the full-information capital structure of the safe type.

The setting in the figure above is the same as in figure 5. The transaction costs parameter \(\kappa\) is fixed at 25 b.p. All firms issue the same type of standardized bond, whose inverse coupon ratio \((p/c)\) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

Risky types will choose to misrepresent themselves if (i) creditors cannot observe their types, and (ii) the MBR differential above is strictly positive. The iso-curves above show that the misrepresentation differential is higher for firms with small to moderate exposure to the volatility shock. Notice however that the impact of the shock size and intensity is not monotonic. The highest MBR differentials are found in the light-colored, triangular region at the bottom, which corresponds to risky types with small, but non-negligible exposure to the shock.
B.2.3 Pooling Payoffs

Figure 10. Safe Type’s Firm Value in an Electronic Platform Pooling Market Outcome

The figure above shows the safe type’s firm value in an Electronic Platform if (i) the safe and risky types were to pool together (choose the same capital structure) and (ii) investors were unable to distinguish one from the other.

The setting is the same as in figure 5. The transaction costs parameter \( \kappa \) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \( (p/c) \) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_h > \sigma_l \) and \( \lambda > 0 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_s = 0.2 \). Because all firms start with low volatility \( \sigma_l = 0.15 \), they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
Figure 11. Safe Type’s Firm Value Differential in an Electronic Platform Pooling Market Outcome

The figure shows the difference between the safe type’s firm value (FV) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

\[ FV_{s}^{\text{POOL}} - FV_{s}^{\text{FI}} \]

where subscript \( s \) stands for “safe”, and superscripts POOL and FI stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 5. The transaction costs parameter \( \kappa \) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio (\( p/c \)) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_h > \sigma_l \) and \( \lambda > 0 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_s = 0.2 \). Because all firms start with low volatility \( \sigma_l = 0.15 \), they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
The figure shows the percentage difference between the safe type’s firm values (FV) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

\[
\frac{FV_{s}\text{POOL} - FV_{s}^{FI}}{FV_{s}^{FI}}
\]

where subscript \( s \) stands for “safe”, and superscripts \( POOL \) and \( FI \) stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 5. The transaction costs parameter \( \kappa \) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \( (p/c) \) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_{h} > \sigma_{l} \) and \( \lambda > 0 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_{s} = 0.2 \). Because all firms start with low volatility \( \sigma_{l} = 0.15 \), they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
Figure 13. Safe Type’s Market-to-Book Ratio Differential in an Electronic Platform Pooling Market Outcome

The figure shows the difference between the safe type’s market-to-book ratios (MBR) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

\[ MBR_{s}^{POOL} - MBR_{s}^{FI} \]

where subscript \( s \) stands for “safe”, and superscripts \( POOL \) and \( FI \) stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 5. The transaction costs parameter \( \kappa \) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \( (p/c) \) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_h > \sigma_l \) and \( \lambda > 0 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_s = 0.2 \). Because all firms start with low volatility \( \sigma_l = 0.15 \), they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
Figure 14. Safe Type’s Market-to-Book Ratio Percentage Differential in an Electronic Platform Pooling Market Outcome

The figure shows the percentage difference between the safe type’s market-to-book ratios (MBR) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

\[
\frac{MBR_{s}^{POOL} - MBR_{s}^{FI}}{MBR_{s}^{FI}}
\]

where subscript \(s\) stands for “safe”, and superscripts \(POOL\) and \(FI\) stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 5. The transaction costs parameter \(\kappa\) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \((p/c)\) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \(\sigma_h > \sigma_l\) and \(\lambda > 0\). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \(\mu_s = 0.2\). Because all firms start with low volatility \(\sigma_l = 0.15\), they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
Figure 15. Risky Type’s Market-to-Book Ratio in an Electronic Platform Pooling Market Outcome

The figure shows the risky type’s market-to-book ratios (MBR) if (i) the safe and risky types were to pool together (choose the same capital structure) and (ii) investors were unable to distinguish one from the other.

The setting is the same as in figure 5. The transaction costs parameter $\kappa$ is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda > 0$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s = 0.2$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
Figure 16. Risky Type’s Market-to-Book Ratio Differential in an Electronic Platform Pooling Market Outcome

The figure shows the difference between the risky type’s market-to-book ratios (MBR) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

\[ MBR_{r,POOL} - MBR_{r,FI} \]

where subscript \( r \) stands for “risky”, and superscripts \( POOL \) and \( FI \) stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 5. The transaction costs parameter \( \kappa \) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \( (p/c) \) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_h > \sigma_l \) and \( \lambda > 0 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_s = 0.2 \). Because all firms start with low volatility \( \sigma_l = 0.15 \), they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
Figure 17. Risky Type’s Market-to-Book Ratio Percentage Differential in an Electronic Platform Pooling Market Outcome

The figure shows the percentage difference between the risky type’s market-to-book ratios (MBR) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

\[
\frac{\text{MBR}^{\text{POOL}}_r - \text{MBR}^{\text{FI}}_r}{\text{MBR}^{\text{FI}}_r}
\]

where subscript \( r \) stands for “risky”, and superscripts \( \text{POOL} \) and \( \text{FI} \) stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 5. The transaction costs parameter \( \kappa \) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \( (p/c) \) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_h > \sigma_l \) and \( \lambda > 0 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_s = 0.2 \). Because all firms start with low volatility \( \sigma_l = 0.15 \), they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
B.2.4 Separating Payoffs

Figure 18. Safe Type’s Firm Value in an Electronic Platform Separating Market Outcome

The figure above shows the safe type’s firm value in an Electronic Platform if safe firms were to adjust their capital structure to ensure separation from risky firms.

The setting is the same as in figure 5. The transaction costs parameter $\kappa$ is fixed at 25 b.p. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda > 0$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s = 0.2$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

$$MBR\left(s_s^{SEP}, s_r^{FI} \mid \gamma = \gamma_r\right) \geq MBR\left(s_s^{SEP}, s_s^{SEP} \mid \gamma = \gamma_r\right)$$

where subscripts $s$ and $r$ correspond to "safe" and "risky", and superscripts SEP and FI stand for "separating" and "full information", respectively.

**Note 1:** No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 8. When this is the case, separating payoffs coincide with those in a full information equilibrium.

**Note 2:** as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
The figure above shows the difference between the safe type's firm value (FV) in a separating market outcome and those payoffs in a full information equilibrium, that is,

$$FV_{SEP} - FV_{FI}$$

where subscript $s$ corresponds to "safe", and superscripts $SEP$ and $FI$ stand for "separating" and "full information", respectively.

The setting is the same as in figure 5. The transaction costs parameter $\kappa$ is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio $(p/c)$ is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda > 0$. Each firm's exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s = 0.2$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

$$MBR(s_{SEP}^{s}, s^{FI}_r | \gamma = \gamma_r) \geq MBR(s_{SEP}^{s}, s^{SEP}_s | \gamma = \gamma_r)$$

**Note 1:** No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 8. When this is the case, separating payoffs coincide with those in a full information equilibrium.

**Note 2:** as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
Figure 20. Safe Type’s Firm Value Percentage Differential in an Electronic Platform Separating Market Outcome

The figure above shows the percentage difference between the safe type’s firm value (FV) in a separating market outcome and those payoffs in a full information equilibrium, that is,

\[
\frac{FV_{s}^{SEP} - FV_{s}^{FI}}{FV_{s}^{FI}}
\]

where subscript \(s\) corresponds to "safe", and superscripts \(SEP\) and \(FI\) stand for "separating" and "full information", respectively.

The setting is the same as in figure 5. The transaction costs parameter \(\kappa\) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \((p/c)\) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \(\sigma_h > \sigma_l\) and \(\lambda > 0\). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \(\mu_s = 0.2\). Because all firms start with low volatility \(\sigma_l = 0.15\), they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

\[
MBR(s_{s}^{SEP}, s_{r}^{FI}|\gamma = \gamma_r) \geq MBR(s_{s}^{SEP}, s_{s}^{SEP}|\gamma = \gamma_r)
\]

Note 1: No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 8. When this is the case, separating payoffs coincide with those in a full information equilibrium.

Note 2: as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
Figure 21. Safe Type’s Market-to-Book Ratio Differential in an Electronic Platform Separating Market Outcome

The figure above shows the difference between the safe type’s market-to-book ratios of equity (MBR) in a separating market outcome and those payoffs in a full information equilibrium, that is,

\[ MBR_{s}^{SEP} - MBR_{s}^{FI} \]

where subscript \(s\) corresponds to "safe", and superscripts \(SEP\) and \(FI\) stand for "separating" and "full information", respectively.

The setting is the same as in figure 5. The transaction costs parameter \(\kappa\) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \(p/c\) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \(\sigma_{h} > \sigma_{l}\) and \(\lambda > 0\). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \(\mu_{s} = 0.2\). Because all firms start with low volatility \(\sigma_{l} = 0.15\), they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

\[ MBR(s_{s}^{SEP}, s_{r}^{FI}|\gamma = \gamma_{r}) \geq MBR(s_{s}^{SEP}, s_{s}^{SEP}|\gamma = \gamma_{r}) \]

Note 1: No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 8. When this is the case, separating payoffs coincide with those in a full information equilibrium.

Note 2: as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
Figure 22. Safe Type’s Market-to-Book Ratio Percentage Differential in an Electronic Platform Separating Market Outcome

The figure above shows the percentage difference between the safe type’s market-to-book ratios of equity (MBR) in a separating market outcome and those payoffs in a full information equilibrium, that is,

\[
\frac{MBR_{s}^{SEP} - MBR_{s}^{FI}}{MBR_{s}^{FI}}
\]

where subscript \( s \) corresponds to "safe", and superscripts \( SEP \) and \( FI \) stand for "separating" and "full information", respectively.

The setting is the same as in figure 5. The transaction costs parameter \( \kappa \) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \((p/c)\) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_h > \sigma_l \) and \( \lambda > 0 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_s = 0.2 \). Because all firms start with low volatility \( \sigma_l = 0.15 \), they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

\[
MBR(s_{s}^{SEP}, s_{r}^{FI}|\gamma = \gamma_r) \geq MBR(s_{s}^{SEP}, s_{s}^{SEP}|\gamma = \gamma_r)
\]

Note 1: No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 8. When this is the case, separating payoffs coincide with those in a full information equilibrium.

Note 2: as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
The figure above shows the safe type’s market-to-book ratios of equity (MBR) in a separating market outcome. By construction, these MBR payoffs coincide with those in a full information equilibrium, since risky types play the full information firm-value-maximizing strategy in a separating market outcome, $s_{r}^{FI}$.

The setting is the same as in figure 5. The transaction costs parameter $\kappa$ is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_{h} > \sigma_{l}$ and $\lambda > 0$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_{s} = 0.2$. Because all firms start with low volatility $\sigma_{l} = 0.15$, they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

$$MBR\left(s_{s}^{SEP}, s_{r}^{FI} | \gamma = \gamma_{r} \right) \geq MBR\left(s_{s}^{SEP}, s_{s}^{SEP} | \gamma = \gamma_{r} \right)$$

Note 1: No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 8. When this is the case, separating payoffs coincide with those in a full information equilibrium.

Note 2: as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
Figure 24. Risky Type’s Market-to-Book Ratio Differential in an Electronic Platform Separating Market Outcome

The figure above shows the difference between the safe type’s market-to-book ratios of equity (MBR) in a separating market outcome and those payoffs in a full information equilibrium, that is,

\[ MBR^\text{SEP}_r - MBR^\text{FI}_r \]

where subscript \( r \) corresponds to "safe", and superscripts \( SEP \) and \( FI \) stand for "separating" and "full information", respectively. By construction, this difference is to zero, since risky types play the full information firm-value-maximizing strategy in a separating market outcome, \( s^\text{FI}_r \).

The setting is the same as in figure 5. The transaction costs parameter \( \kappa \) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \( (p/c) \) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_h > \sigma_l \) and \( \lambda > 0 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_s = 0.2 \). Because all firms start with low volatility \( \sigma_l = 0.15 \), they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

\[ MBR\left(s^\text{SEP}_s, s^\text{FI}_r | \gamma = \gamma_r\right) \geq MBR\left(s^\text{SEP}_s, s^\text{SEP}_s | \gamma = \gamma_r\right) \]

Note 1: No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 8. When this is the case, separating payoffs coincide with those in a full information equilibrium.

Note 2: as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
Figure 25. Risky Type’s Market-to-Book Ratio Percentage Differential in an Electronic Platform Separating Market Outcome

The figure above shows the percentage difference between the safe type’s market-to-book ratios of equity (MBR) in a separating market outcome and those payoffs in a full information equilibrium, that is,

\[ (MBR_{SEP}^{r} - MBR_{FI}^{r}) \times (MBR_{r}^{FI})^{-1} \]

where subscript \( r \) corresponds to "safe", and superscripts \( SEP \) and \( FI \) stand for "separating" and "full information", respectively. By construction, this difference is to zero, since risky types play the full information firm-value-maximizing strategy in a separating market outcome, \( s_{r}^{FI} \).

The setting is the same as in figure 5. The transaction costs parameter \( \kappa \) is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \( (p/c) \) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_{h} > \sigma_{l} \) and \( \lambda > 0 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_{s} = 0.2 \). Because all firms start with low volatility \( \sigma_{l} = 0.15 \), they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

\[ MBR(s_{s}^{SEP}, s_{r}^{FI} | \gamma = \gamma_{r}) \geq MBR(s_{s}^{SEP}, s_{s}^{SEP} | \gamma = \gamma_{r}) \]

Note 1: No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 8. When this is the case, separating payoffs coincide with those in a full information equilibrium.

Note 2: as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
The figure above shows the safe type’s firm value in equilibrium when the only secondary market is an Electronic Platform. The light gray area on the LHS plot indicates the set of risky types for which a pooling equilibrium would prevail. Above and to the left of this area, a separating equilibrium holds. When the exposure to the volatility shock is sufficiently high, misrepresentation does not benefit the risky firms’ shareholders, so the equilibrium under asymmetric information coincides with full-information equilibrium.

The setting is the same as in figure 5. The transaction costs parameter $\kappa$ is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds. In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_r > \sigma_s$ and $\lambda > 0$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s = 0.2$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors.

In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its initial firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types not benefiting from misrepresentation. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is: $MBR(s^{SEP}_s, s^{SEP}_r | \gamma = \gamma_r) \geq MBR(s^{SEP}_s, s^{SEP}_s | \gamma = \gamma_r)$, where subscripts $s$ and $r$ correspond to "safe" and "risky", and superscripts SEP and FI stand for "separating" and "full information", respectively. The equilibrium market outcome is that which yields the highest safe-type’s initial firm valuation.
Figure 27. Risky Type’s Market-to-Book Ratio of Equity in Equilibrium in an Electronic Platform

The figure above shows the risky type’s MBR in equilibrium when the only secondary market is an Electronic Platform. The light gray area on the LHS plot indicates the set of risky types for which a pooling equilibrium would prevail. Above and to the left of this area, a separating equilibrium holds. When the exposure to the volatility shock is sufficiently high, misrepresentation does not benefit the risky firms’ shareholders, so the equilibrium under asymmetric information coincides with full-information equilibrium.

The setting is the same as in figure 5. The transaction costs parameter $\kappa$ is fixed at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose in a full information equilibrium if it were free to issue non-standardized bonds. In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda > 0$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s = 0.2$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors.

In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its initial firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types not benefiting from misrepresentation. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

$$MBR(s_{s}^{SEP}, s_{r}^{FI} | \gamma = \gamma_r) \geq MBR(s_{s}^{SEP}, s_{s}^{SEP} | \gamma = \gamma_r),$$

where subscripts $s$ and $r$ correspond to "safe" and "risky", and superscripts $SEP$ and $FI$ stand for "separating" and "full information", respectively. The equilibrium market outcome is that which yields the highest safe-type’s initial firm valuation.
B.2.6 Dual Market Equilibria

Figure 28. Safe Type’s Firm Value in a Dual-Market Equilibrium

The figure above shows the safe type’s initial firm valuation when firms have the option to issue standardized bonds to be traded in EP, or add a debt protective covenant that fully reveals their type, but restricts secondary trades to the less liquid OTC market. Safe firms issue standardized debt whenever the cross-market liquidity differential exceeds the informational costs of adverse selection in electronic exchanges, as defined in section VI.

The setting and iso-curve levels are the same as in figure 26, except that now firms can issue non-standardized debt to be traded in an OTC market with transaction cost parameter $\kappa^{OTC} = 32.5$ b.p.. As before, the abbreviations POOL, SEP and FI stand for pooling, separating and full-information equilibria in electronic platform. The light gray area (OTC) on the LHS plot indicates the set of risky types that would prompt safe firms to issue non-standardized bonds, leading to a separating equilibrium in which debt from safe firms is traded over the counter, while risky firms’ bonds are transacted in electronic platforms. Notice how the levels of the iso-curves in this region are below the safe firm’s valuation in OTC, $FV^{OTC} = 111.41$. Down and to the left, the grey area (POOL) corresponds to the set of risky types for which a pooling equilibrium in EP would prevail. Above and to the left of the dual market equilibrium area, a separating equilibrium holds (regions SEP and FI.) As before, when the exposure to the volatility shock is sufficiently high, misrepresentation does not benefit the risky firms’ shareholders, so the equilibrium under asymmetric information coincides with full-information equilibrium (FI.)

In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its initial firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types not benefitting from misrepresentation. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is: $MBR(s_s^{SEP}, s_r^{FI} | \gamma = \gamma_r) \geq MBR(s_s^{SEP}, s_r^{SEP} | \gamma = \gamma_r)$, where subscripts $s$ and $r$ correspond to "safe" and "risky", and superscripts $SEP$ and $FI$ stand for "separating" and "full information", respectively. The equilibrium market outcome is that which yields the highest safe-type’s initial firm valuation. A dual market equilibrium prevails if safe firms can increase their value by issuing non-standardized bonds with a debt protective covenant that fully reveals their type.
B.3 Contour Plots - $\sigma_h \times \mu_s$ Plots

B.3.1 Full Information Equilibrium

**Figure 29.** Optimal Firm Values in an Electronic Platform Full Information Equilibrium

The figure above shows the optimal firm values in an Electronic Platform full information equilibrium for risky firms of varying shock size exposure, $\sigma_h$, and different measures of entrant safe type firms, $\mu_s$. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E. Since creditors differentiate between types, the firm valuations are independent from the measure of safe firms. Therefore, the iso curves levels above match the cross-section of the firm value function in figure 5 along $\lambda = 0.3$. 
**Figure 30.** Optimal Market-to-Book Ratios in an Electronic Platform Full Information Equilibrium

The figure above shows the optimal market-to-book ratios in an Electronic Platform full information equilibrium for risky firms of varying shock size exposure, $\sigma_h$, and different measures of entrant safe type firms, $\mu_s$. The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E. Since creditors differentiate between types, the MBR’s are independent from the measure of safe firms. Therefore, the iso curves levels above match the cross-section of the MBR function in figure 6 along $\lambda = 0.3$. 
B.3.2 Misrepresentation

Figure 31. Misrepresentation Market-to-Book Ratios in an Electronic Platform

The figure shows the market-to-book ratios of equity (MBR) in an Electronic Platform for risky firms of varying shock size exposure, $\sigma_h$, and different measures of entrant safe type firms, $\mu_s$, if these firms misrepresented themselves by copying the full-information capital structure of the safe type.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio $(p/c)$ is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

Risky types will choose to misrepresent themselves if (i) creditors cannot observe their types, and (ii) the MBR yielded by copying the capital structure of the safe type exceeds their optimal MBR.

Since, by construction, the misrepresenting payoffs are computed by having creditors assign probability 1 to the event that bonds are issued by safe firms, the misrepresenting MBR’s are independent from the measure of safe firms. Therefore, the iso curves levels above match the cross-section of the misrepresenting MBR function in figure 7 along $\lambda = 0.3$. 
**Figure 32.** Risky Type’s MBR Differential in EP - Misrepresentation v.s. Full Information Equilibrium Payoffs

The figure shows the difference in market-to-book ratios (MBR) yielded by a type-misrepresentation and a truth-telling (Full Information) strategies in an Electronic Platform, that is,

\[ MBR_{r}^{MP} - MBR_{r}^{FI} \]

where subscript \( r \) stands for “risky”, and superscripts \( MP \) and \( FI \) stand for “misrepresentation” and “full information”, respectively. The differences are computed for varying levels of shock size exposure, \( \sigma_h \), and different measures of entrant safe type firms, \( \mu_s \). The type-misrepresentation strategy consists in copying the full-information capital structure of the safe type.

The setting is the same as in figure 29. The volatility shock intensity is fixed at \( \lambda = 0.3 \), while the transaction costs parameter \( \kappa \) is kept at 25 b.p. All firms issue the same type of standardized bond, whose inverse coupon ratio ( \( p/c \)) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

Risky types will choose to misrepresent themselves if (i) creditors cannot observe their types, and (ii) the MBR yielded by copying the capital structure of the safe type exceeds their optimal MBR.

Since, by construction, the misrepresenting payoffs are computed by having creditors assign probability 1 to the event that bonds are issued by safe firms, the misrepresenting MBR’s are independent from the measure of safe firms. Therefore, the iso curves levels above match the cross-section of the misrepresenting MBR function in figure 8 along \( \lambda = 0.3 \).
**Figure 33.** Risky Type’s MBR Percentage Differential in EP - Misrepresentation v.s. Full Information Equilibrium Payoffs

The figure shows the percentage difference in market-to-book ratios (MBR) yielded by a type-misrepresentation and a truth-telling (Full Information) strategies in an Electronic Platform, that is,

\[
\frac{MBR_{MP}^r - MBR_{FI}^r}{MBR_{FI}^r}
\]

where subscript \( r \) stands for “risky”, and superscripts \( MP \) and \( FI \) stand for “misrepresentation” and “full information”, respectively. The differences are computed for varying levels of shock size exposure, \( \sigma_h \), and different measures of entrant safe type firms, \( \mu_s \). The type-misrepresentation strategy consists in copying the full-information capital structure of the safe type.

The setting is the same as in figure 29. The volatility shock intensity is fixed at \( \lambda = 0.3 \), while the transaction costs parameter \( \kappa \) is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ( \( p/c \)) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

Risky types will choose to misrepresent themselves if (i) creditors cannot observe their types, and (ii) the MBR yielded by copying the capital structure of the safe type exceeds their optimal MBR.

Since, by construction, the misrepresenting payoffs are computed by having creditors assign probability 1 to the event that bonds are issued by safe firms, the misrepresenting MBR’s are independent from the measure of safe firms. Therefore, the iso curves levels above match the cross-section of the misrepresenting MBR function in figure 9 along \( \lambda = 0.3 \).
### B.3.3 Pooling Payoffs

**Figure 34.** Safe Type’s Firm Value in an Electronic Platform Pooling Market Outcome

![Safe Type’s Optimal Firm Value in a Pooling Equilibrium](image)

The figure above shows the safe type’s firm value in an Electronic Platform if (i) the safe and risky types were to pool together (choose the same capital structure) and (ii) investors were unable to distinguish one from the other.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio $(p/c)$ is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda = 0.3$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

*Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.*
Figure 35. Safe Type’s Firm Value Differential in an Electronic Platform Pooling Market Outcome

The figure shows the difference between the safe type’s firm value (FV) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

$$FV_{s}^{POOL} - FV_{s}^{FI}$$

where subscript $s$ stands for “safe”, and superscripts $POOL$ and $FI$ stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda = 0.3$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
Figure 36. Safe Type’s Firm Value Percentage Differential in an Electronic Platform Pooling Market Outcome

The figure shows the percentage difference between the safe type’s firm values (FV) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

$$\frac{FV_s^{POOL} - FV_s^{FI}}{FV_s^{FI}}$$

where subscript s stands for “safe”, and superscripts POOL and FI stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio $(\rho/c)$ is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda = 0.3$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
Figure 37. Safe Type’s Market-to-Book Ratio Differential in an Electronic Platform Pooling Market Outcome

The figure shows the difference between the safe type’s market-to-book ratios (MBR) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

$$MBR_{sa}^{POOL} - MBR_{sa}^{FI}$$

where subscript $s$ stands for “safe”, and superscripts $POOL$ and $FI$ stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda = 0.3$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
Figure 38. Safe Type’s Market-to-Book Ratio Percentage Differential in an Electronic Platform Pooling Market Outcome

The figure shows the percentage difference between the safe type’s market-to-book ratios (MBR) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

\[
\frac{MBR_{s}^{POOL} - MBR_{s}^{FI}}{MBR_{s}^{FI}}
\]

where subscript \( s \) stands for “safe”, and superscripts \( POOL \) and \( FI \) stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 29. The volatility shock intensity is fixed at \( \lambda = 0.3 \), while the transaction costs parameter \( \kappa \) is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio (\( p/c \)) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_{h} > \sigma_{l} \) and \( \lambda = 0.3 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_{s} \). Because all firms start with low volatility \( \sigma_{l} = 0.15 \), they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
**Figure 39.** Risky Type’s Market-to-Book Ratio in an Electronic Platform Pooling Market Outcome

The figure shows the risky type’s market-to-book ratios (MBR) if (i) the safe and risky types were to pool together (choose the same capital structure) and (ii) investors were unable to distinguish one from the other.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda = 0.3$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types' probability.

*Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.*
Figure 40. Risky Type’s Market-to-Book Ratio Differential in an Electronic Platform Pooling Market Outcome

The figure shows the difference between the risky type’s market-to-book ratios (MBR) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

$$MBR_{r}^{POOL} - MBR_{r}^{FI}$$

where subscript \( r \) stands for “risky”, and superscripts \( POOL \) and \( FI \) stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 29. The volatility shock intensity is fixed at \( \lambda = 0.3 \), while the transaction costs parameter \( \kappa \) is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio (\( p/c \)) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_{h} > \sigma_{l} \) and \( \lambda = 0.3 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_{s} \). Because all firms start with low volatility \( \sigma_{l} = 0.15 \), they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
The figure shows the percentage difference between the risky type’s market-to-book ratios (MBR) when types pool together (choose the same capital structure) and those payoffs in a full information equilibrium, that is,

$$\frac{MBR^{POOL}_r - MBR^{FI}_r}{MBR^{FI}_r}$$

where subscript $r$ stands for “risky”, and superscripts $POOL$ and $FI$ stand for “pooling” and “full information”, respectively.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda = 0.3$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Finally, bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability.

Note: as explained in the text, whether a pooling equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than the corresponding value in a separating equilibrium.
### B.3.4 Separating Payoffs

**Figure 42.** Safe Type’s Firm Value in an Electronic Platform Separating Equilibrium

The figure above shows the safe type’s firm value in an Electronic Platform if safe firms were to adjust their capital structure to ensure separation from risky firms. Since in a separating market outcome creditors differentiate between types, the payoffs above are independent from the measure of safe firms.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda = 0.3$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

$$MBR_s(\sigma_s^{SEP}, \sigma_r^{FI} | \gamma = \gamma_r) \geq MBR_s(\sigma_s^{SEP}, \sigma_s^{SEP} | \gamma = \gamma_r)$$

where subscripts $s$ and $r$ correspond to "safe" and "risky", and superscripts $SEP$ and $FI$ stand for "separating" and "full information", respectively.

**Note1:** No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 32. When this is the case, separating payoffs coincide with those in a full information equilibrium.

**Note2:** as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
Figure 43. Safe Type’s Firm Value Differential in an Electronic Platform Separating Equilibrium

The figure above shows the difference between the safe type’s firm value (FV) in a separating market outcome and those payoffs in a full information equilibrium, that is,

$$FV_{s}^{SEP} - FV_{s}^{FI}$$

where subscript $s$ corresponds to "safe", and superscripts $SEP$ and $FI$ stand for "separating" and "full information", respectively. Since in a separating market outcome creditors differentiate between types, the payoffs above are independent from the measure of safe firms.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda = 0.3$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

$$MBR(s_{s}^{SEP}, s_{r}^{FI} | \gamma = \gamma_r) \geq MBR(s_{s}^{SEP}, s_{s}^{SEP} | \gamma = \gamma_r)$$

where subscripts $s$ and $r$ correspond to "safe" and "risky", and superscripts $SEP$ and $FI$ stand for "separating" and "full information", respectively.

Note1: No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 32. When this is the case, separating payoffs coincide with those in a full information equilibrium.

Note2: as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
Figure 44. Safe Type’s Firm Value Percentage Differential in an Electronic Platform Separating Equilibrium

The figure above shows the percentage difference between the safe type’s firm value (FV) in a separating market outcome and those payoffs in a full information equilibrium, that is,

\[
\frac{FV^\text{SEP}_s - FV^\text{FI}_s}{FV^\text{FI}_s}
\]

where subscript \(s\) corresponds to "safe", and superscripts \(\text{SEP}\) and \(\text{FI}\) stand for "separating" and "full information", respectively. Since in a separating market outcome creditors differentiate between types, the payoffs above are independent from the measure of safe firms.

The setting is the same as in figure 29. The volatility shock intensity is fixed at \(\lambda = 0.3\), while the transaction costs parameter \(\kappa\) is kept at 25 b.p. All firms issue the same type of standardized bond, whose inverse coupon ratio \((p/c)\) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \(\sigma_h > \sigma_l\) and \(\lambda = 0.3\). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \(\mu_s\). Because all firms start with low volatility \(\sigma_l = 0.15\), they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

\[
MBR\left(s^\text{SEP}_s, s^\text{SEP}_r | \gamma = \gamma_r\right) \geq MBR\left(s^\text{SEP}_s, s^\text{FI}_r | \gamma = \gamma_r\right)
\]

where subscripts \(s\) and \(r\) correspond to "safe" and "risky", and superscripts \(\text{SEP}\) and \(\text{FI}\) stand for "separating" and "full information", respectively.

**Note1:** No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 32. When this is the case, separating payoffs coincide with those in a full information equilibrium.

**Note2:** as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
Figure 45. Safe Type’s Market-to-Book Ratio of Equity Differential in an Electronic Platform Separating Equilibrium

The figure above shows the difference between the safe type’s market-to-book ratios of equity (MBR) in a separating market outcome and those payoffs in a full information equilibrium, that is,

\[ MBR^{SEP}_s - MBR^{FI}_s \]

where subscript \( s \) corresponds to "safe", and superscripts \( SEP \) and \( FI \) stand for "separating" and "full information", respectively. Since in a separating market outcome creditors differentiate between types, the payoffs above are independent from the measure of safe firms.

The setting is the same as in figure 29. The volatility shock intensity is fixed at \( \lambda = 0.3 \), while the transaction costs parameter \( \kappa \) is kept at 25 b.p. All firms issue the same type of standardized bond, whose inverse coupon ratio \( (p/c) \) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_h > \sigma_l \) and \( \lambda = 0.3 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_s \). Because all firms start with low volatility \( \sigma_l = 0.15 \), they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

\[ MBR\left(s^{SEP}_s, s^{FI}_r | \gamma = \gamma_r \right) \geq MBR\left(s^{SEP}_s, s^{SEP}_s | \gamma = \gamma_r \right) \]

where subscripts \( s \) and \( r \) correspond to "safe" and "risky", and superscripts \( SEP \) and \( FI \) stand for "separating" and "full information", respectively.

Note 1: No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 32. When this is the case, separating payoffs coincide with those in a full information equilibrium.

Note 2: as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
Figure 46. Safe Type’s Market-to-Book Ratio of Equity Percentage Differential in an Electronic Platform Separating Equilibrium

The figure above shows the percentage difference between the safe type’s market-to-book ratios of equity (MBR) in a separating market outcome and those payoffs in a full information equilibrium, that is,

$$\frac{MBR^S_{\text{SEP}} - MBR^S_{\text{FI}}}{MBR^S_{\text{FI}}}$$

where subscript $s$ corresponds to "safe", and superscripts SEP and FI stand for "separating" and "full information", respectively. Since in a separating market outcome creditors differentiate between types, the payoffs above are independent from the measure of safe firms.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda = 0.3$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

$$MBR\left(s^S_{\text{SEP}}, s^S_{r, \gamma = \gamma_r} \right) \geq MBR\left(s^S_{\text{SEP}}, s^S_{s, \gamma = \gamma_r} \right)$$

where subscripts $s$ and $r$ correspond to "safe" and "risky", and superscripts SEP and FI stand for "separating" and "full information", respectively.

**Note1:** No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 32. When this is the case, separating payoffs coincide with those in a full information equilibrium.

**Note2:** as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
The figure above shows the safe type’s market-to-book ratios of equity (MBR) in a separating market outcome. By construction, these MBR payoffs coincide with those in a full information equilibrium, since risky types play the full information firm-value-maximizing strategy in a separating market outcome, $s_r^{FI}$. Moreover, since in a separating market outcome creditors differentiate between types, the payoffs above are independent from the measure of safe firms.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda = 0.3$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

$$\text{MBR} \left(s_s^{SEP}, s_r^{FI} \mid \gamma = \gamma_r\right) \geq \text{MBR} \left(s_s^{SEP}, s_s^{SEP} \mid \gamma = \gamma_r\right)$$

where subscripts $s$ and $r$ correspond to "safe" and "risky", and superscripts $SEP$ and $FI$ stand for "separating" and "full information", respectively.

Note1: No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 32. When this is the case, separating payoffs coincide with those in a full information equilibrium.

Note2: as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
Figure 48. Risky Type’s Market-to-Book Ratio of Equity Differential in an Electronic Platform Separating Market Outcome

The figure above shows the difference between the safe type's market-to-book ratios of equity (MBR) in a separating market outcome and those payoffs in a full information equilibrium, that is,

$$MBR^{SEP}_{s} - MBR^{FI}_{r}$$

where subscript $r$ corresponds to "safe", and superscripts $SEP$ and $FI$ stand for "separating" and "full information", respectively. By construction, this difference is to zero, since risky types play the full information firm-value-maximizing strategy in a separating market outcome, $s^{FI}_{r}$.

The setting is the same as in figure 29. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p. All firms issue the same type of standardized bond, whose inverse coupon ratio ($p/c$) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_{h} > \sigma_{l}$ and $\lambda = 0.3$. Each firm's exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_{s}$. Because all firms start with low volatility $\sigma_{l} = 0.15$, they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors' funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

$$MBR(s^{SEP}_{s}, s^{FI}_{r} | \gamma = \gamma_{r}) \geq MBR(s^{SEP}_{s}, s^{SEP}_{s} | \gamma = \gamma_{r})$$

where subscripts $s$ and $r$ correspond to "safe" and "risky", and superscripts $SEP$ and $FI$ stand for "separating" and "full information", respectively.

Note1: No adjustment to the safe type's capital structure is necessary if the risky type's MBR differential is negative in figure 32. When this is the case, separating payoffs coincide with those in a full information equilibrium.

Note2: as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
**Figure 49.** Risky Type’s Market-to-Book Ratio of Equity Percentage Differential in an Electronic Platform Separating Market Outcome

![Graph showing the percentage difference between the safe type’s market-to-book ratios of equity (MBR) in a separating market outcome and those payoffs in a full information equilibrium.](image)

The figure above shows the percentage difference between the safe type’s market-to-book ratios of equity (MBR) in a separating market outcome and those payoffs in a full information equilibrium, that is,

\[
\frac{MBR_{SEP} - MBR_{FI}}{MBR_{FI}}
\]

where subscript \( r \) corresponds to "safe", and superscripts \( SEP \) and \( FI \) stand for "separating" and "full information", respectively. By construction, this difference is to zero, since risky types play the full information firm-value-maximizing strategy in a separating market outcome, \( s_{FI} \).

The setting is the same as in figure 29. The volatility shock intensity is fixed at \( \lambda = 0.3 \), while the transaction costs parameter \( \kappa \) is kept at 25 b.p. All firms issue the same type of standardized bond, whose inverse coupon ratio \( (p/c) \) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E.

In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_h > \sigma_l \) and \( \lambda = 0.3 \). Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_s \). Because all firms start with low volatility \( \sigma_l = 0.15 \), they are ex-ante identical to investors. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types opting not to pool together by issuing the same amount of debt. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is:

\[
MBR\left(s_{SEP}^{SEP}, s_{r,FI}^{FI} | \gamma = \gamma_r \right) \geq MBR\left(s_{SEP}^{SEP}, s_{SEP}^{SEP} | \gamma = \gamma_r \right)
\]

where subscripts \( s \) and \( r \) correspond to "safe" and "risky", and superscripts \( SEP \) and \( FI \) stand for "separating" and "full information", respectively.

**Note 1:** No adjustment to the safe type’s capital structure is necessary if the risky type’s MBR differential is negative in figure 32. When this is the case, separating payoffs coincide with those in a full information equilibrium.

**Note 2:** as explained in the text, whether a separating equilibrium prevails depends on the firm value of the safe type under this market arrangement being greater than that in a pooling equilibrium.
The figure above shows the safe type's firm value in equilibrium when the only secondary market is an Electronic Platform. The light gray area on the LHS plot indicates the set of risky types for which a pooling equilibrium would prevail. To the left of this area, a separating equilibrium holds. When the exposure to the volatility shock is sufficiently high, misrepresentation does not benefit the risky firms' shareholders, so the equilibrium under asymmetric information coincides with full-information equilibrium.

The setting is the same as in figure 29. The volatility shock intensity is fixed at \( \lambda = 0.3 \), while the transaction costs parameter \( \kappa \) is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio \( (p/c) \) is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E. In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which \( \sigma_h > \sigma_l \) and \( \lambda = 0.3 \). Each firm's exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, \( \mu_s \). Because all firms start with low volatility \( \sigma_l = 0.15 \), they are ex-ante identical to investors.

In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its initial firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types not benefiting from misrepresentation. By the creditors' funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is: 

\[
MBR(s_{s,SEP}^{SEP}, r_{s,SEP}) \geq MBR(s_{s,SEP}^{SEP}, r_{s,SEP}^{SEP} | \gamma = \gamma_r),
\]

where subscripts \( s \) and \( r \) correspond to "safe" and "risky", and superscripts SEP and FI stand for "separating" and "full information", respectively. The equilibrium market outcome is that which yields the highest safe-type's initial firm valuation.
The figure above shows the risky type’s MBR in equilibrium when the only secondary market is an Electronic Platform. The light gray area on the LHS plot indicates the set of risky types for which a pooling equilibrium would prevail. To the left of this area, a separating equilibrium holds. When the exposure to the volatility shock is sufficiently high, misrepresentation does not benefit the risky firms’ shareholders, so the equilibrium under asymmetric information coincides with full-information equilibrium.

The setting is the same as in figure 50. The volatility shock intensity is fixed at $\lambda = 0.3$, while the transaction costs parameter $\kappa$ is kept at 25 b.p.. All firms issue the same type of standardized bond, whose inverse coupon ratio $(p/c)$ is set to the nearest half-integer of the optimal inverse coupon ratio that the safe type would optimally choose when following the approach described in section II.E. In computing these payoffs, only two types of firms are considered at a time: the safe type and a risky type, for which $\sigma_h > \sigma_l$ and $\lambda = 0.3$. Each firm’s exposure to the volatility shock is observable by its shareholders alone. However, creditors do know the time-invariant distribution of types, which is fully characterized by the measure of safe firms, $\mu_s$. Because all firms start with low volatility $\sigma_l = 0.15$, they are ex-ante identical to investors.

In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its initial firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types not benefiting from misrepresentation. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is: $\text{MBR}(s_s^{SEP}, s_r^{FI} | \gamma = \gamma_r) \geq \text{MBR}(s_s^{SEP}, s_s^{SEP} | \gamma = \gamma_r)$, where subscripts $s$ and $r$ correspond to "safe" and "risky", and superscripts $SEP$ and $FI$ stand for "separating" and "full information", respectively. The equilibrium market outcome is that which yields the highest safe-type’s initial firm valuation.
B.3.6 Dual Market Equilibria

Figure 52. Safe Type’s Firm Value in a Dual-Market Equilibrium

Safe Type’s Optimal Firm Value in the Prevailing Dual Market Equilibria
for $m = 1.00$, $p/c = 12.0$, $\xi = 1.00$, $\kappa_{EP}(b.p.) = 25.00$, $\lambda = 0.300$, $\sigma_l = 0.150$, $\kappa_{OTC}(b.p.) = 32.50$

Safe Type’s Full Information Firm Value = 111.81, Safe Type’s OTC Firm Value = 111.41

The figure above shows the safe type’s initial firm valuation when firms have the option to issue standardized bonds to be traded in EP, or add a debt protective covenant that fully reveals their type, but restricts secondary trades to the less liquid OTC market. Safe firms issue standardized debt whenever the cross-market liquidity differential exceeds the informational costs of adverse selection in electronic exchanges, as defined in section VI.

The setting and iso-curve levels are the same as in figure 26, except that now firms can issue non-standardized debt to be traded in an OTC market with transaction cost parameter $\kappa_{OTC} = 32.5$ b.p. As before, the abbreviations POOL, SEP and FI stand for pooling, separating and full-information equilibria in electronic platform. The light gray area (OTC) on the LHS plot indicates the combinations of risk exposure and $\mu_s$ that would prompt safe firms to issue non-standardized bonds, leading to a separating equilibrium in which debt from safe firms is traded over the counter, while risky firms’ bonds are transacted in electronic platforms. Notice how the levels of the iso-curves in this region are below the safe firm’s valuation in OTC, $FV_{sOTC} = 111.41$. Above the OTC region, the grey area (POOL) corresponds to the set of risky types for which a pooling equilibrium in EP would prevail. To the left of the dual market and pooling equilibrium areas, a separating equilibrium holds (regions SEP and FI.) As before, when the exposure to the volatility shock is sufficiently high, misrepresentation does not benefit the risky firms’ shareholders, so the equilibrium under asymmetric information coincides with full-information equilibrium (FI.)

In a pooling market outcome, the safe type chooses the amount of outstanding bonds that maximizes its initial firm value, conditional on risky-types opting to pool together by issuing the same amount of debt. Bond prices are computed as a sum of the type-conditional bond prices, weighted by each types’ probability. In a separating market outcome, the safe type chooses the amount of outstanding bonds that maximizes its firm value, conditional on risky-types not benefiting from misrepresentation. By the creditors’ funding condition, risky firms play the (full information equilibrium) firm-value-maximizing strategy, so their incentive compatibility constraint is: $MBR(s_{SEP}^s, s_{SEP}^r \mid \gamma = \gamma_r) \geq MBR(s_{SEP}^s, s_{SEP}^s \mid \gamma = \gamma_r)$, where subscripts $s$ and $r$ correspond to "safe" and "risky", and superscripts SEP and FI stand for "separating" and "full information", respectively. The equilibrium market outcome is that which yields the highest safe-type’s initial firm valuation. A dual market equilibrium prevails if safe firms can increase their value by issuing non-standardized bonds with a debt protective covenant that fully reveals their type.
Appendix C. First-Passage Time Distribution

C.1 Dynamics

Consider the following value processes with geometric brownian motion dynamics in the risk-neutral measure:

\[
\frac{dV_i}{V_i} = r_{\text{grow}} dt + \sigma_i dZ_t
\]

where \( r_{\text{grow}} > 0 \) is the growth rate and \( i = l, h \). The solution to this SDE is

\[
V_s = V_t \exp \left( \left( r_{\text{grow}} - \frac{1}{2} \sigma_i^2 \right) (s - t) + \sigma_i (Z_s - Z_t) \right)
\]

for \( s > t \).

Define the normalized process as

\[
v_i^t \equiv \ln \left( \frac{V_i^t}{V_i^B} \right)
\]

Then \( v^i \) is a brownian motion in the risk-neutral measure satisfying:

\[
dv_i^t = \left( r_{\text{grow}} - \frac{1}{2} \sigma_i^2 \right) dt + \sigma_i dZ_t
\]

C.2 Some Notation

Denote by \( r_{\text{disc}} \geq 0 \) the investor’s rate of discount, and let \( \sigma \in \{ \sigma_l, \sigma_h \} \). The following definitions will be used throughout the derivations. Let \( \theta_0 \equiv (r_{\text{grow}}, \sigma) \) and \( \theta \equiv (r_{\text{disc}}, \theta_0) = (r_{\text{disc}}, r_{\text{grow}}, \sigma) \).

- Parameters

\[
\hat{a} = \hat{a} (\theta_0) \equiv \frac{r_{\text{grow}} - \frac{1}{2} \sigma^2}{\sigma^2}
\]

\[
\hat{z} = \hat{z} (\theta) \equiv \frac{\hat{a} (\theta_0)^2 \sigma^4 + 2 r_{\text{disc}} \sigma^2}{\sigma^2}
\]

- Functions

\[
h_1 (u, v; \theta_0) \equiv \frac{-v - \hat{a} (\theta_0) \sigma^2 u}{\sigma \sqrt{u}}, \quad h_2 (u, v; \theta_0) \equiv \frac{-v + \hat{a} (\theta_0) \sigma^2 u}{\sigma \sqrt{u}}
\]

\[
q_1 (u, v; \theta) \equiv \frac{-v - \hat{z} (\theta) \sigma^2 u}{\sigma \sqrt{u}}, \quad q_2 (u, v; \theta) \equiv \frac{-v + \hat{z} (\theta) \sigma^2 u}{\sigma \sqrt{u}}
\]

\[
\Psi (u, v; \theta_0) \equiv N \left( -h_1 (u, v; \theta_0) \right) - e^{-2 \hat{a} (\theta_0) v} N (h_2 (u, v; \theta_0))
\]

\[
F (u, v; \theta_0) \equiv 1 - \Psi (u, v; \theta_0) = N \left( h_1 (u, v; \theta_0) \right) + e^{-2 \hat{a} (\theta_0) v} N (h_2 (u, v; \theta_0))
\]

\[
G (u, v; \theta) \equiv e^{(\hat{z} (\theta) - \hat{a} (\theta_0)) v} N (q_1 (u, v; \theta)) + e^{-(\hat{z} (\theta) + \hat{a} (\theta_0)) v} N (q_2 (u, v; \theta))
\]

where \( N (\cdot) \) is the cumulative normal distribution function.
C.3 Distribution of the the Time of Default

**Lemma 1:** For a fixed volatility level \( \sigma \), define \( t^d \) as the first time \( v \) hits 0. Conditional on solvency of the firm up to period \( t > 0 \), the probability of default occurring after period \( s > t \) is given by

\[
P \left\{ t^d > s \mid \mathcal{F}_t \right\} = \Psi \left( s - t, v_t; \theta_0 \right) = N \left( h_1 \left( s - t, v_t; \theta_0 \right) \right) - e^{-2\hat{\alpha}(\theta_0)v_t} N \left( h_2 \left( s - t, v_t; \theta_0 \right) \right)
\]

where, \( v_t \equiv \ln \left( \frac{V_t}{V^B(\sigma)} \right), h_1 \left( u, v; \theta_0 \right), h_2 \left( u, v; \theta_0 \right) \) and \( \hat{\alpha} \left( \theta_0 \right) \) are as defined above. Denoting by \( F \left( s, v_t; \theta_0 \right) \) the cumulative distribution of the default stopping time, we get

\[
F \left( s, v_t; \theta_0 \right) = P \left( \hat{\tau} \leq s \mid \mathcal{F}_t \right) = N \left( h_1 \left( s - t, v_t; \theta_0 \right) \right) + e^{-2\hat{\alpha}(\theta_0)v_t} N \left( h_2 \left( s - t, v_t; \theta_0 \right) \right)  
\]

(C1)

**Proof.** Let \( X \) be a \( (r_{\text{grow}} - \frac{1}{2}\sigma^2, \sigma) \) brownian motion on a given probability space \((\Omega, \mathcal{F}, \mathbb{P})\):

\[
X_u \equiv \left( r_{\text{grow}} - \frac{1}{2} \sigma^2 \right) u + \sigma Z_u
\]

It follows that

\[
dX_u = \hat{\alpha}(\theta_0) \sigma^2 du + \sigma dZ_u
\]

Define

\[
m_t^V \equiv \inf_{t \leq u \leq s} V_u,
\]

\[
m_t^X \equiv \inf_{0 \leq u \leq s} X_u
\]

and let \( v_t \equiv \ln \left( \frac{V_t}{V^B(\sigma)} \right) \). Then,

\[
P \left\{ t^d > s \mid \mathcal{F}_t \right\} = P \left( \inf \left\{ u \geq t : V_u \leq V^B(\sigma) \right\} > s \mid \mathcal{F}_t \right) = P \left( \inf_{0 \leq u \leq s-t} X_{t+u} > -v_t \mid \mathcal{F}_t \right) = P \left( m_{s-t}^X > -v_t \mid \mathcal{F}_t \right)
\]

By Lemma 3.1.2 in Bielecki and Rutkowski (2004), this probability is equal to

\[
P \left\{ m_{s-t}^X > -\ln \left( \frac{V_t}{V^B(\sigma)} \right) \mid \mathcal{F}_t \right\} = N \left( \frac{v_t + \hat{\alpha}(\theta_0) \sigma^2 (s - t)}{\sigma \sqrt{s - t}} \right) - e^{-2\hat{\alpha}(\theta_0)v_t} N \left( \frac{-v_t + \hat{\alpha}(\theta_0) \sigma^2 (s - t)}{\sigma \sqrt{s - t}} \right)
\]

Defining the first-passage time cumulative distribution as

\[
F \left( s, v_t; \theta_0 \right) = P \left( t^d \leq s \mid \mathcal{F}_t \right)
\]

we obtain

\[
F \left( s, v_t; \theta_0 \right) = 1 - P \left( t^d > s \mid \mathcal{F}_t \right) = 1 - \Psi \left( s - t, v_t; \theta_0 \right) = 1 - N \left( -h_1 \left( s - t, v_t; \theta_0 \right) \right) + e^{-2\hat{\alpha}(\theta_0)v_t} N \left( h_2 \left( s - t, v_t; \theta_0 \right) \right)
\]

\[
\Rightarrow F \left( s, v_t \right) = N \left( h_1 \left( s - t, v_t; \theta_0 \right) \right) + e^{-2\hat{\alpha}(\theta_0)v_t} N \left( h_2 \left( s - t, v_t; \theta_0 \right) \right)
\]

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Therefore,
\[
F(s, v_t; \theta_0) = 1 - \Psi(s - t, v_t; \theta_0) = N(h_1(s - t, v_t; \theta_0)) + e^{-2\hat{\alpha}(\theta_0)v}N(h_2(s - t, v_t; \theta_0))
\]

COROLLARY 1: The probability density function of the default stopping time is
\[
P\left(t^d \in ds | \mathcal{F}_t \right) = -\Psi'(s - t, v_t; \theta_0) = N'(h_1(s - t, v_t; \theta_0)) \frac{v_t}{\sigma (s - t)^{3/2}}
\]

Proof. We start by noticing that
\[
\frac{\partial h_{1,2}(u, v; \theta_0)}{\partial \tau} = \frac{\partial h_{\pm}(u, v; \theta_0)}{\partial \tau} = \frac{\mp\hat{\alpha}(\theta_0) \sigma}{\sqrt{u}} - \frac{1}{2u}h_{1,2}(u, v; \theta_0)
\]
and
\[
e^{-2\hat{\alpha}(\theta_0)v}N'(h_2(s - t, v_t; \theta_0)) = N'(h_1(s - t, v_t; \theta_0))
\]
The probability density function is obtained by differentiating \(F(s, v_t; \theta_0)\) with respect to time:
\[
P\left(t^d \in ds | \mathcal{F}_t \right) = -\Psi_1(s - t, v_t; \theta_0)
\]
\[
= N'(h_1(s - t, v_t; \theta_0)) \frac{\partial h_1(s - t, v_t; \theta_0)}{\partial \tau}
\]
\[
+ e^{-2\hat{\alpha}(\theta_0)v}N'(h_2(s - t, v_t; \theta_0)) \frac{\partial h_2(s - t, v_t; \theta_0)}{\partial \tau}
\]
where subscript 1 denotes the derivative of \(\Psi\) with respect to its first argument.

Plugging in the formulas in equations \(C2\) and \(C3\), we obtain
\[
P\left(t^d \in ds | \mathcal{F}_t \right) = N'(h_1(s - t, v_t; \theta_0)) \frac{v_t}{\sigma (s - t)^{3/2}}
\]

C.4 Joint Probability Law of \(V\) and \(t^d\)

LEMMA 2: The joint probability law of \(V\) and the default stopping time \(t^d\) is given by
\[
P\left(V_s \geq V; t^d \geq s | \mathcal{F}_t \right) = \Psi^{v,t^d}(s - t, \ln(V/V_B(\sigma)); \ln(V_t/V_B(\sigma)), \theta_0)
\]
where
\[
\Psi^{v,t^d}(u, \sigma; v, \theta_0) = N\left(-\frac{-\overline{v} + v + \hat{\alpha}(\theta_0)\sigma^2u}{\sqrt{u}} \right) - e^{-2\hat{\alpha}(\theta_0)v}N\left(-\frac{-\overline{v} + v + \hat{\alpha}(\theta_0)\sigma^2u}{\sqrt{u}} \right)
\]
\[
= N\left(-\frac{-\overline{v}}{\sqrt{u}} + h_1(u, v; \theta_0) \right) - e^{-2\hat{\alpha}(\theta_0)v}N\left(-\frac{-\overline{v}}{\sqrt{u}} + h_2(u, v; \theta_0) \right)
\]

Proof. Using the definition of \(m^X_s\) from lemma 1, we can express the joint law of \(V\) and \(t^d\) by
\[
P\left(V_s \geq V; t^d \geq s | \mathcal{F}_t \right) = P\left(V_s \geq V; \inf \{u \geq t: V_u \leq V_B(\sigma)\} > s \right) \left| \mathcal{F}_t \right.
\]
\[
= P\left(X_{s-t} \geq -\ln\left(\frac{V_t}{V} \right); m^X_{s-t} > -\ln\left(\frac{V_t}{V_B(\sigma)} \right) \right) \left| \mathcal{F}_t \right.
\]

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for \( V \geq V_B(\sigma) \). By Lemma 3.1.3 in Bielecki and Rutkowski (2004), we obtain

\[
P \left( V_s \geq V; t^d \geq s \mid \mathcal{F}_t \right) = N \left( \frac{\ln \left( \frac{V}{\bar{V}} \right) + \hat{\alpha}(\theta_0) \sigma^2 (s - t)}{\sigma \sqrt{s - t}} \right)
\]

\[
- e^{2\hat{\alpha}(\theta_0) \left( -\ln \left( \frac{V}{\bar{V}} \right) \right)} \cdot N \left( \frac{-2\ln \left( \frac{V}{\bar{V}} \right) + \ln \left( \frac{V}{\bar{V}} \right) + \hat{\alpha}(\theta_0) \sigma^2 (s - t)}{\sigma \sqrt{s - t}} \right)
\]

\[
= N \left( \frac{-\ln \left( \frac{V}{\bar{V}} \right) + \ln \left( \frac{V}{\bar{V}} \right) + \hat{\alpha}(\theta_0) \sigma^2 (s - t)}{\sigma \sqrt{s - t}} \right)
\]

It follows from equations C5 and C6 that

\[
P \left( V_s \geq V; t^d \geq s \mid \mathcal{F}_t \right) = \Psi^{v,t^d}(s - t, \ln \left( \frac{V}{V_B(\sigma)} \right); \ln \left( \frac{V_t}{V_B(\sigma)} \right), \theta_0)
\]

(C7)

**COROLLARY 2:** The conditional probability density function of \( V \) is

\[
P \left( V_s \in dV \mid \mathcal{F}_t; t^d \geq s \right) = -\frac{1}{\bar{V}} \cdot f^{v,t^d} \left( s - t, \ln \left( \frac{V}{V_B(\sigma)} \right); \ln \left( \frac{V_t}{V_B(\sigma)} \right), \theta_0 \right)
\]

where

\[
f^{v,t^d}(u, \bar{v}; v, \theta_0) = -\frac{1}{\Psi(u, v, \theta_0)} \frac{\partial}{\partial v} \Psi^{v,t^d}(u, \bar{v}; v, \theta_0)
\]

\[
= \frac{1}{\sigma \sqrt{u}} \left\{ N' \left( \frac{-\bar{v}}{\sigma \sqrt{u}} - h_1(u, v; \theta_0) \right) - e^{-2\hat{\alpha}(\theta_0) \bar{v}} N' \left( \frac{-\bar{v}}{\sigma \sqrt{u}} + h_2(u, v; \theta_0) \right) \right\}
\]

**Proof.** By Lemmas 1 and 2, a direct application of Bayes’ Rule gives

\[
P \left( V_s \geq V \mid \mathcal{F}_t; t^d \geq s \right) = \frac{P \left( V_s \geq V; t^d \geq s \mid \mathcal{F}_t \right)}{P \left( t^d \geq s \mid \mathcal{F}_t \right)}
\]

\[
= \frac{\Psi^{v,t^d}(s - t, \ln \left( \frac{V}{V_B(\sigma)} \right); \ln \left( \frac{V_t}{V_B(\sigma)} \right), \theta_0)}{\Psi \left( s - t, \ln \left( \frac{V_t}{V_B(\sigma)} \right), \theta_0 \right)}
\]
Therefore,

\[
P(\Psi_s \leq V \mid \Phi_t; t^d \geq s) = 1 - \frac{\Psi^{\nu, \xi_d}(s - t, \ln \left( \frac{V}{V^B(\sigma)} \right); \ln \left( \frac{V_i}{V^B(\sigma)} \right), \theta_0)}{\Psi(s - t, \ln \left( \frac{V_i}{V^B(\sigma)} \right), \theta_0)}
\]

The conditional probability density function is obtained by taking the derivative of the expression above with respect to \( V \):

\[
P(\Psi_s \in dV \mid \Phi_t; t^d \geq s) = -\frac{1}{\Psi(s - t, \ln \left( \frac{V_i}{V^B(\sigma)} \right), \theta_0)} \cdot \frac{\partial}{\partial V} \Psi^{\nu, \xi_d}(s - t, \ln \left( \frac{V}{V^B(\sigma)} \right); \ln \left( \frac{V_i}{V^B(\sigma)} \right), \theta_0)
\]

\[
= -\frac{1}{\Psi(s - t, \ln \left( \frac{V_i}{V^B(\sigma)} \right), \theta_0)} \left\{ -\ln \left( \frac{V}{V^B(\sigma)} \right) - h_1(s - t, v_1; \theta_0) \right\} \left[ -\frac{1}{\sigma \sqrt{s - t}} \right] - e^{-2\alpha(\theta_0)v_1} N'(\frac{-\ln \left( \frac{V}{V^B(\sigma)} \right) + h_2(s - t, v_1; \theta_0)}{\sigma \sqrt{s - t}}) \left[ -\frac{1}{\sigma \sqrt{s - t}} \right]
\]

Define the function \( f^{\nu, \xi_d}(u, v; \theta_0) \) as

\[
f^{\nu, \xi_d}(u, v; \theta_0) \equiv -\frac{1}{\Psi(u, v; \theta_0)} \frac{\partial}{\partial u} \Psi^{\nu, \xi_d}(u, v; \theta_0)
\]

\[
= -\frac{1}{\sigma \sqrt{u}} \left\{ N'(\frac{v}{\sigma \sqrt{u}} - h_1(u, v; \theta_0)) - e^{-2\alpha(\theta_0)v} N'(\frac{-v}{\sigma \sqrt{u}} + h_2(u, v; \theta_0)) \right\}
\]

Then

\[
P(\Psi_s \in dV \mid \Phi_t; t^d \geq s) = -\frac{1}{V} \cdot f^{\nu, \xi_d}(s - t, \ln \left( \frac{V}{V^B(\sigma)} \right); \ln \left( \frac{V_i}{V^B(\sigma)} \right), \theta_0)
\]

\[\square\]

\section*{C.5 An Important Result}

In what follows, we shall make use of the formula below.

\textbf{CLAIM 2:} Let \( a, b, c \in \mathbb{R} \) satisfy \( b < 0 \) and \( c^2 > a \). Then for every \( y > 0 \)

\[
\int_0^y e^{ay} dN \left( \frac{b - cx}{\sqrt{x}} \right) = \frac{d + c}{2d} g_1(y) + \frac{d - c}{2d} g_2(y)
\]

where \( d = \sqrt{c^2 - 2a} \) and

\[
g_1(y) = e^{b \left( c - d \right)} N \left( \frac{b - dy}{\sqrt{y}} \right), \quad g_2(y) = e^{b \left( c + d \right)} N \left( \frac{b + dy}{\sqrt{y}} \right)
\]


\[\square\]

\textbf{PROPOSITION 1:} The discounted integral of the default stopping time probability density function is equal to

\[
\int_t^{t+}\! \! e^{-r \text{disc}(u-t)} \Psi_1(u-t, v_1) du = -G(\tau, v_1; \theta) \quad (C8)
\]
where $\theta = (r_{\text{disc}}, \theta_0)$ and $G$ is as defined in Appendix C.2:

$$
G (u, v; \theta) \equiv e^{(\hat{\xi}(\theta) - \hat{\alpha}(\theta_0))v N (q_1 (u, v; \theta)) + e^{-(\hat{\xi}(\theta) + \hat{\alpha}(\theta_0))v N (q_2 (u, v; \theta))}
$$

for

$$
q_1 (u, v; \theta) \equiv \left( -v - \hat{\xi} (\theta) \sigma^2 u \right) / \sigma \sqrt{u}, \quad q_2 (u, v; \theta) \equiv \left( -v + \hat{\xi} (\theta) \sigma^2 u \right) / \sigma \sqrt{u}
$$

**Proof.** Let $v_t \equiv \ln (V_t / V^B)$, for some arbitrary boundary $V^B$. We have

$$
\int_t^{t + \tau} e^{-r_{\text{disc}}(u - t)} \Psi_1 \left( u - t, \ln \left( \frac{V_t}{V^B} \right); \theta_0 \right) du = - \int_t^{t + \tau} e^{-r_{\text{disc}}(u - t)} F_1 \left( u - t, \ln \left( \frac{V_t}{V^B} \right) \right) du
$$

$$
= - \left\{ \int_t^{t + \tau} e^{-r_{\text{disc}}(u - t)} dN \left( -v_t - \hat{\alpha} (\theta_0) \sigma^2 (u - t) \right) / \sigma \sqrt{u - t} \right\} + e^{-2 \hat{\alpha} (\theta_0) v_t} \int_t^{t + \tau} e^{-r_{\text{disc}}(u - t)} dN \left( -v_t + \hat{\alpha} (\theta_0) \sigma^2 (u - t) \right) / \sigma \sqrt{u - t} \}
$$

Changing variables, let $x = u - t$. Then $du = dt$ and $t \leq u \leq t + \tau$ is equivalent to $0 \leq x \leq \tau$. Therefore,

$$
\int_t^{t + \tau} e^{-r_{\text{disc}}(u - t)} F_1 \left( u - t, v_t \right) du = - \int_0^{\tau} e^{-r_{\text{disc}}x} dN \left( \frac{-1}{\sigma} v_t - \hat{\alpha} (\theta_0) \sigma x \right) / \sqrt{x}
$$

$$
+ e^{-2 \hat{\alpha} (\theta_0) v_t} \int_0^{\tau} e^{-r_{\text{disc}}x} dN \left( \frac{-1}{\sigma} v_t + \hat{\alpha} (\theta_0) \sigma x \right) / \sqrt{x}
$$

The result follows from application of the formula in claim 2 to the expression above. The algebra is somewhat tedious and is thus omitted here.

### Appendix D. Building Blocks

#### D.1 The Value of Primitive Claims

In what follows, let $\mathcal{F}$ be the filtration associated to the stochastic process governing the value of the firm’s assets $V$ under a given volatility regime ($\sigma = \sigma_i, i = l, h$). Denote by $t^\sigma$ the time of the first occurrence of a volatility shock, and define the right-continuous process $H_t \equiv 1_{\{t^\sigma \leq t\}}$. To account for the arrival of volatility shocks, expectations are computed with respect to the augmented filtration $\mathcal{G} \equiv \mathcal{F} \vee \mathcal{H}$, where $\mathcal{H} \equiv (\mathcal{H}_t)_{t \geq 0}$, with $\mathcal{H}_t \equiv \sigma (H_u, u \leq t)$.

**LEMMA 3:** [The value of a claim that pays $1 continuously up to the first occurrence of a volatility shock] Let $r_{\text{disc}}$ be the investors’ rate of discount, and let $t^\sigma$ denote the first arrival time of a volatility shock. Denote by $p^\sigma_t \equiv p^\sigma (H_t)$ the value of a claim that pays $1 continuously up to the first occurrence of a volatility shock. Then

$$
p^\sigma_t = E^Q \left[ \int_t^{t^\sigma} e^{-r_{\text{disc}}(u - t)} \cdot 1 du \bigg| \mathcal{G}_t \right]
$$

$$
= 1_{\{t^\sigma > t\}} \frac{1}{r_{\text{disc}} + \lambda}
$$

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Proof. We have

\[
p_t^\sigma = E^Q \left[ \int_t^{\tau^\sigma} e^{-r_{\text{disc}}(u-t)} \cdot 1du \bigg| \mathcal{F}_t \right]
\]
\[
= E^Q \left[ \int_t^{\infty} e^{-r_{\text{disc}}(u-t)} (1 - H_u) du \bigg| \mathcal{F}_t \right]
\]
\[
= \int_t^{\infty} e^{-r_{\text{disc}}(u-t)} E^Q \left[ 1_{\{t^\sigma > u\}} \bigg| \mathcal{F}_t \right] du
\]

By independence of \(H\) and \(V\),

\[
E^Q \left[ 1_{\{t^\sigma > u\}} \bigg| \mathcal{F}_t \right] = E^Q \left[ 1_{\{t^\sigma > u\}} \bigg| \mathcal{H}_t \right]
\]

Since a volatility shock arrives according to a Poisson process with intensity \(\lambda\),

\[
E^Q \left[ 1 - H_u \bigg| \mathcal{H}_t \right] = E^Q \left[ 1_{\{t^\sigma > u\}} \bigg| \mathcal{H}_t \right]
\]
\[
= 1_{\{t^\sigma > t\}} P \left( t^\sigma > u \bigg| \mathcal{F}_t \right)
\]
\[
= 1_{\{t^\sigma > t\}} e^{-\lambda(u-t)}
\]

(D1)

It follows that

\[
p_t^\sigma = 1_{\{t^\sigma > t\}} \int_t^{\infty} e^{-(r_{\text{disc}}+\lambda)(u-t)} du = 1_{\{t^\sigma > t\}} \frac{1}{r_{\text{disc}} + \lambda}
\]

\[\square\]

**LEMMA 4:** [The Value of a Claim that Pays $1 at Maturity in the Absence of Volatility Shocks and Default] Let \(r_{\text{disc}}\) be the investors’ rate of discount, and let \(t^\sigma\) denote the first arrival time of a volatility shock. Denote by \(p^{zc}(V_t, \tau; V^B, \theta)\) the value of zero-coupon bond that pays $1 at time \(t + \tau\) in the absence of a volatility shock and default. Then

\[
p^{zc}(V_t, \tau; V^B, \theta) = E^Q \left[ e^{-r_{\text{disc}} \tau^\sigma} 1_{\{t^\sigma \wedge \tau > t + \tau\}} \bigg| \mathcal{F}_t \right]
\]
\[
= 1_{\{t^\sigma \wedge \tau > t\}} e^{-(r_{\text{disc}}+\lambda)\tau} \Psi (\tau, v_t; \theta_0)
\]

where \(\Psi\) is as defined in Appendix C.2.

**Proof.** Consider

\[
p^{zc}(V_t, \tau; V^B, \theta) = E^Q \left[ e^{-r_{\text{disc}} \tau^\sigma} 1_{\{t^\sigma \wedge \tau > t + \tau\}} \bigg| \mathcal{F}_t \right]
\]
\[
= e^{-r_{\text{disc}} \tau} E^Q \left[ 1_{\{t^\sigma > t + \tau\}} \cdot 1_{\{t^\sigma > t + \tau\}} \bigg| \mathcal{F}_t \right]
\]

By independence of \(H\) and \(V\),

\[
E^Q \left[ 1_{\{t^\sigma > t + \tau\}} \cdot 1_{\{t^\sigma > t + \tau\}} \bigg| \mathcal{F}_t \right] = E^Q \left[ 1_{\{t^\sigma > t + \tau\}} \bigg| \mathcal{H}_t \right] E^Q \left[ 1_{\{t^\sigma > t + \tau\}} \bigg| \mathcal{F}_t \right]
\]
\[
= 1_{\{t^\sigma > t\}} P \left( t^\sigma > t + \tau \bigg| \mathcal{H}_t \right) \cdot 1_{\{t^\sigma > t\}} P \left( t^\sigma > t + \tau \bigg| \mathcal{F}_t \right)
\]
\[
= 1_{\{t^\sigma \wedge \tau > t\}} e^{-\lambda \tau} P \left( t^\sigma > t + \tau \bigg| \mathcal{F}_t \right)
\]

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Finally, by Lemma 1, we get

\[ p^{zc} (V_t, \tau; V^B, \theta) = 1_{\{t_d \land \tau > t\}} e^{-(r_{disc} + \lambda) \tau} \Psi (\tau, v_t; \theta_0) \]

\[ \square \]

**Lemma 5:** [The Value of a Claim that Pays $1 if Default Happens Before Maturity and Volatility Shock] Let \( r_{disc} \) be the investors’ rate of discount, \( t_d \) the time of default, and \( t^* \) the first arrival time of a volatility shock. Denote by \( p^{\text{default}} (V_t, \tau; V^B, \theta) \) the value of a claim that pays $1 in case default happens before maturity and volatility shock \( (t_d < t^* \land t + \tau) \), where . Then

\[ p^{\text{default}} (V_t, \tau; V^B, \theta) = 1_{\{t_d \land \tau > t\}} E^Q \left[ e^{-r_{disc} (t^* - t)} 1_{\{t^* < t_d \land t + \tau\}} \right] \]

\[ = 1_{\{t_d \land \tau > t\}} G (\tau, \ln \left( \frac{V_t}{V^B} \right) ; (r_{disc} + \lambda, r_{grow}, \sigma)) \]

where \( G \) is as defined in Appendix C.2. Notice that the volatility shock hazard rate is added to the investors’ discount rate \( r_{disc} \).

**Proof.** Expanding the expectation term on the RHS, we get

\[ p^{\text{default}} (V_t, \tau; V^B, \theta) = E^Q \left[ e^{-r_{disc} (t^* - t)} 1_{\{t^* > t_d \}} \right] \]

\[ = E^Q \left[ e^{-r_{disc} (t^* - t)} 1_{\{t^* > t_d \}} \cdot 1_{\{t_d < t + \tau\}} \right] \]

\[ = \int_{t}^\infty e^{-r_{disc} (s - t)} E^Q \left[ 1_{\{t^* > s\}} \cdot 1_{\{t_d < t + \tau\}} \cdot 1_{\{t_d \land \tau > t\}} \right] ds \]

\[ = \int_{t}^{t^* + \tau} e^{-r_{disc} (s - t)} E^Q \left[ 1_{\{t^* > s\}} \cdot 1_{\{t_d \land \tau > t\}} \right] ds \]

By independence of \( H \) and \( V \) and Corollary 1, the expectation inside the integral becomes

\[ E^Q \left[ 1_{\{t^* > s\}} \cdot 1_{\{t_d \land \tau > t\}} \right] = E^Q \left[ 1_{\{t^* > s\}} \right] E^Q \left[ 1_{\{t_d \land \tau > t\}} \right] \]

\[ = 1_{\{t^* > t_d\}} P \left( t^* > s \left| H^t \right. \right) \left[ -1_{\{t^* > t_d\}} \Psi_1 (s - t, \ln \left( \frac{V_t}{V^B} \right) ; \theta_0) \right] \]

\[ = -1_{\{t_d \land \tau > t\}} e^{-\lambda (s - t)} \Psi_1 (s - t, \ln \left( \frac{V_t}{V^B} \right) ; \theta_0) \]

which gives

\[ p^{\text{default}} (V_t, \tau; V^B, \theta) = -1_{\{t_d \land \tau > t\}} \int_{t}^{t^* + \tau} e^{-(r_{disc} + \lambda)(s - t)} \Psi_1 (s - t, \ln \left( \frac{V_t}{V^B} \right) ; \theta_0) ds \]

Finally, the result follows from direct application of equation C8 in Claim 2:

\[ p^{\text{default}} (V_t, \tau; V^B, \theta) = 1_{\{t_d \land \tau > t\}} G (\tau, \ln \left( \frac{V_t}{V^B} \right) ; (r_{disc} + \lambda, r_{grow}, \sigma)) \]

\[ \square \]
LEMMA 6: [The Value of a Claim that Pays $1 Continuously Until Maturity in the Absence of Volatility Shocks and Default] Let $r_{\text{disc}}$ be the investors' rate of discount, $t^d$ the time of default, and $t^s$ the first arrival time of a volatility shock. Denote by $p_{\text{strips}}(V_t, \tau; V^B, \theta)$ the value of a claim that pays $1$ continuously up to time $t + \tau$ in the absence of a volatility shock and default. Then

$$p_{\text{strips}}(V_t, \tau; V^B, \theta) = E^Q \left[ \int_t^{t+\tau} e^{-r_{\text{disc}}(u-t)} 1_{\{t^d \land \tau^s > u\}} du \bigg| \mathcal{F}_t \right]$$

$$= 1_{\{t^d \land \tau^s > t\}} \frac{1}{r_{\text{disc}} + \lambda} \left[ 1 - e^{-(r_{\text{disc}} + \lambda)^\tau} \Psi (\tau, v_t; \theta_0) \right] -$$

$$- 1_{\{t^d \land \tau^s > t\}} \frac{1}{r_{\text{disc}} + \lambda} G \left( \tau, \ln \left( \frac{V_t}{V^B} \right) ; (r_{\text{disc}} + \lambda, r_{\text{grow}}, \sigma) \right)$$

**Proof.** Consider

$$p_{\text{strips}}(V_t, \tau; V^B, \theta) = E^Q \left[ \int_t^{t+\tau} e^{-r_{\text{disc}}(u-t)} 1_{\{t^d \land \tau^s > u\}} du \bigg| \mathcal{F}_t \right]$$

$$= \int_t^{t+\tau} e^{-r_{\text{disc}}(u-t)} E^Q \left[ 1_{\{t^d \land \tau^s > u\}} \bigg| \mathcal{F}_t \right] du$$

$$= \int_t^{t+\tau} E^Q \left[ e^{-r_{\text{disc}}(u-t)} 1_{\{t^d \land \tau^s > u\}} \bigg| \mathcal{F}_t \right] du$$

By Lemma 4, the RHS is equal to

$$p_{\text{strips}}(V_t, \tau; V^B, \theta) = \int_t^{t+\tau} p_{\text{zc}}(V_t, u-t; V^B, \theta) du$$

$$= 1_{\{t^d \land \tau^s > t\}} \int_t^{t+\tau} e^{-(r_{\text{disc}} + \lambda)(u-t)} \Psi (u-t, v_t; \theta_0) du$$

Integrating by parts, we obtain

$$p_{\text{strips}}(V_t, \tau; V^B, \theta) = 1_{\{t^d \land \tau^s > t\}} \left( - \frac{e^{-(r_{\text{disc}} + \lambda)(u-t)}}{r_{\text{disc}} + \lambda} \Psi (u-t, v_t; \theta_0) \right) \bigg|_{u=t}^{t+\tau}$$

$$- 1_{\{t^d \land \tau^s > t\}} \int_t^{t+\tau} \left( - \frac{e^{-(r_{\text{disc}} + \lambda)(u-t)}}{r_{\text{disc}} + \lambda} \right) \Psi_1 (u-t, v_t; \theta_0) du$$

$$= 1_{\{t^d \land \tau^s > t\}} \frac{1}{r_{\text{disc}} + \lambda} \left[ \Psi (0, v_t; \theta_0) e^{-r_{\text{disc}} \tau} \Psi (\tau, v_t; \theta_0) \right]$$

$$+ 1_{\{t^d \land \tau^s > t\}} \frac{1}{r_{\text{disc}} + \lambda} \int_t^{t+\tau} e^{-(r_{\text{disc}} + \lambda)(u-t)} \Psi_1 (u-t, v_t; \theta_0) du$$

By formula C8 in Claim 2, the last integral term on the RHS equals $-G \left( \tau, \ln \left( \frac{V_t}{V^B} \right) ; (r_{\text{disc}} + \lambda, r_{\text{grow}}, \sigma) \right)$. Therefore,

$$p_{\text{strips}}(V_t, \tau; V^B, \theta) = 1_{\{t^d \land \tau^s > t\}} \frac{1}{r_{\text{disc}} + \lambda} \left[ 1 - e^{-(r_{\text{disc}} + \lambda)^\tau} \Psi (\tau, v_t; \theta_0) \right]$$

$$- 1_{\{t^d \land \tau^s > t\}} \frac{1}{r_{\text{disc}} + \lambda} G \left( \tau, \ln \left( \frac{V_t}{V^B} \right) ; (r_{\text{disc}} + \lambda, r_{\text{grow}}, \sigma) \right)$$

□
Notice we can break down the value of the claim in Lemma 6 in three parts:

\[ p_{\text{strips}}(V_t, \tau; V^B, \theta) = 1_{\{d > t\}} \left( \frac{1}{r_{\text{disc}} + \lambda} - 1_{\{d \wedge \tau > t\}} \frac{1}{r_{\text{disc}} + \lambda} \frac{e^{-(r_{\text{disc}} + \lambda)\tau}}{\Psi (\tau, v_0; \theta_0)} \right) \]

Value of a claim to \( p^c_t \) at time \( t + \tau \) if \( t^a \wedge t^d > t + \tau \)

\[ = p^c_t \times p^c(V_t, \tau; V^B, \theta) \]

Therefore,

\[ p_{\text{strips}}(V_t, \tau; V^B, \theta) = p^c_t \left\{ 1_{\{d > t\}} - p^c(V_t, \tau; V^B, \theta) - p^\text{default}(V_t, \tau; V^B, \theta) \right\} \quad (D2) \]

where \( \theta_0 = (r_{\text{grow}}, \sigma) \) and \( \theta = (r_{\text{disc}}, r_{\text{grow}}, \sigma) \).

**Lemma 7**: The Value of a Claim that Pays a V-Contingent Bond Payoff when the Volatility Shock Happens Before Default and Before Maturity

Let \( r_{\text{disc}} \) be the investors’ rate of discount, \( t^d \) the time of default, and \( t^a \) the first arrival time of a volatility shock. Denote by \( p^{\sigma, \text{bond}}(V_t, \tau; \theta) \) the value of a claim that pays \( d_\sigma(V_t, \tau - t^a; V^B(\sigma_h), \theta_h) \) when initial volatility is \( \sigma_t \) and the volatility shock happens before maturity \( \tau \) and before default.

\[ p^{\sigma, \text{bond}}(V_t, \tau) \equiv E^Q \left[ e^{-r_{\text{disc}}(t^a - t)} 1_{\{t^a < t^d \wedge (t + \tau)\}} d_\sigma(V_t, \tau - (t^a - t); V^B(\sigma_h), \theta_h) \big| F_t \right] \]

Then the value of \( p^{\sigma, \text{bond}}(V_t, \tau, \theta_t) \) can be computed as

\[ p^{\sigma, \text{bond}}(V_t, \tau, \theta_t) = -1_{\{t^a \wedge \tau > t\}} \int_t^{t^a} e^{-(r_{\text{disc}} + \lambda)(s - t)} \left\{ \int_{V^B(\sigma_t)}^{\infty} d_\sigma(V_t, \tau - (s - t); V^B(\sigma_h), \theta_h) \times \frac{\partial}{\partial V} \Psi^{v, t^d} \left( s - t, \ln \left( \frac{V}{V^B(\sigma_t)} \right), \ln \left( \frac{V}{V^B(\sigma_h)} \right), \theta_{0, t} \right) dV \right\} ds \]

where \( \theta_{0, t} = (r_{\text{grow}}, \sigma_t) \) and \( \theta_i = (r_{\text{disc}}, r_{\text{grow}}, \sigma_i) \), for \( i \in \{t, \tau\} \), and \( \Psi^{v, t^d} \) is given by equation C6.

**Proof**. For notational convenience, in what follows let \( d_h(V, \tau) \equiv d_\sigma(V, \tau; V^B(\sigma_h), \theta_h) \). Expressing the term inside the expectation in the expression for \( p^{\sigma, \text{bond}}(V_t, \tau, \theta_t) \) in integral form, we obtain

\[ e^{-r_{\text{disc}}(t^a - t)} 1_{\{t^a < t^d \wedge (t + \tau)\}} d_h(V_t, \tau - (t^a - t)) = \int_t^{t^a} e^{-r_{\text{disc}}(s - t)} d_h(V, \tau - (s - t)) 1_{\{t^d < s < t^a \}} ds \]

\[ = \int_t^{t^a} e^{-r_{\text{disc}}(s - t)} d_h(V, \tau - (s - t)) 1_{\{t^d < s\}} 1_{\{s < t^a\}} ds \]

This allows us to rewrite \( p^{\sigma, \text{bond}}(V_t, \tau; \theta_t) \) as

\[ p^{\sigma, \text{bond}}(V_t, \tau; \theta_t) = \int_t^{t^a} e^{-r_{\text{disc}}(s - t)} E^Q \left[ d_h(V, \tau - (s - t)) 1_{\{t^d < s\}} 1_{\{s < t^a\}} \big| F_t \right] ds \]
By independence of \( t^d_l \) and \( t^\sigma \) (or, equivalently, \( V \) and \( H \)),
\[
E^Q \left[ d_h(V, \tau - (s-t)) 1_{\{t^d_l > s\}} 1_{\{t^\sigma \in ds\}} \right] = E^Q \left[ d_h(V, \tau - (s-t)) 1_{\{t^d_l > s\}} \right] E^Q \left[ 1_{\{t^\sigma \in ds\}} \right] \\
= E^Q \left[ d_h(V, \tau - (s-t)) 1_{\{t^d_l > s\}} \right] \mathcal{F}_t E^Q \left[ 1_{\{t^\sigma \in ds\}} \right] \\
\]
Therefore,
\[
p^{\sigma, \text{bond}} (V_t, \tau; \theta_l) = \int_t^{t+\tau} e^{-r_{\text{disc}}(s-t)} E^Q \left[ d_h(V, \tau - (s-t)) 1_{\{t^d_l > s\}} \right] \left( 1_{\{t^\sigma > t\}} \lambda e^{-\lambda(s-t)} \right) \cdot ds \\
= 1_{\{t^\sigma > t\}} \lambda \int_t^{t+\tau} e^{-(r_{\text{disc}}+\lambda)(s-t)} E^Q \left[ d_h(V, \tau - (s-t)) 1_{\{t^d_l > s\}} \right] \mathcal{F}_t \cdot ds \quad (D3)
\]
Application of Bayes’ rules to the remaining expectation term above gives
\[
E^Q \left[ d_h(V, \tau - (s-t)) 1_{\{t^d_l > s\}} \right] \mathcal{F}_t = \int_{V^B(\sigma_l)}^\infty d_h(V, \tau - (s-t)) \left[ 1_{\{t^d_l > t\}} P \left( V_s \in dV; t^d_l \geq s \middle| \mathcal{F}_t \right) \right] dV \\
= 1_{\{t^d_l > t\}} \int_{V^B(\sigma_l)}^\infty d_h(V, \tau - (s-t)) \frac{P \left( V_s \in dV; t^d_l \geq s \middle| \mathcal{F}_t \right)}{P \left( t^d_l \geq s \middle| \mathcal{F}_t \right)} dV \quad (D4)
\]
By Lemma 1 and Corollary 2, the fraction on the RHS equals
\[
- \frac{\partial}{\partial V} \Psi^{v,t^d_l} \left( s-t, \ln \left( \frac{V}{V^B(\sigma_l)} \right); \ln \left( \frac{V_t}{V^B(\sigma_l)} \right), \theta_{0,l} \right)
\]
Plugging the formula for \( d_h(V, \tau) \) and the expression above into equation (D4), we obtain
\[
E^Q \left[ d_\P (V, \tau; V^B(\sigma_h), \theta_h) 1_{\{t^d_l > s\}} \right] \mathcal{F}_t = -1_{\{t^d_l > t\}} \int_{V^B(\sigma_l)}^\infty d_\P (V, \tau - (s-t); V^B(\sigma_h), \theta_h) \times \\
\times \frac{\partial}{\partial V} \Psi^{v,t^d_l} \left( s-t, \ln \left( \frac{V}{V^B(\sigma_l)} \right); \ln \left( \frac{V_t}{V^B(\sigma_l)} \right), \theta_{0,l} \right) dV \quad (D5)
\]
The result follows from equations (D3) and (D5).

\[\square\]

**D.2 Preliminary Laplace Transform Results**

Define the Laplace transformation of a function \( E(v) \) as
\[
F(s) \equiv L[E(v)] = \int_0^\infty e^{-sv} E(v) \, dv \quad (D6)
\]
for some \( s > 0 \). Then, for \( \beta < s \),
\[
L \left[ e^{\beta v} \right] = \frac{1}{s-\beta} \quad (D7)
\]
Moreover, for \( p > -s \),
\[
L \left[ e^{-pvN ((p-x)\sigma \sqrt{m})} \right] = \frac{1}{s+p} N \left( (p-x)\sigma \sqrt{m} \right) \quad (D8)
\]

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CLAIM 3: The Laplace Transform of a Bond Pricing Function Term

\[
L \left[ e^{-\beta v} N \left( \frac{-v - \alpha\sigma^2 u}{\sigma \sqrt{u}} \right) \right] = \frac{1}{s + \beta} \left\{ N \left( -\alpha\sigma\sqrt{u} \right) + e^{\frac{1}{2}\left[(s+(\alpha+\beta))^2-\alpha^2\right]}\sigma^2 u N \left( - (s + (\alpha + \beta)) \sigma \sqrt{u} \right) \right\}
\]

(D9)

Proof. Let \( \alpha \in \mathbb{R}^* \) and define

\[
h(u,v) \equiv \frac{-v - \alpha\sigma^2 u}{\sigma \sqrt{u}}
\]

For \( \beta > -s \), define

\[
\kappa \equiv s + \beta
\]

Then

\[
L \left[ e^{-\beta v} N \left( h(u,v) \right) \right] = \int_0^\infty e^{-(s + \beta)v} N(h(u,v)) \, dv
\]

\[
= \left( -\frac{e^{-\kappa v}}{\kappa} N(h(u,v)) \right) \bigg|_{v=0}^{v=\infty} - \int_0^\infty \left( \frac{1}{\kappa} e^{-\kappa v} \right) dN(h(u,v))
\]

\[
= \frac{1}{\kappa} N(h(u,0)) + \frac{1}{\kappa} \int_0^\infty e^{-\kappa v} N\left( \frac{-v - \alpha\sigma^2 u}{\sigma \sqrt{u}} \right) \left( -\frac{1}{\sigma \sqrt{u}} \right) \, dv
\]

\[
= \frac{1}{\kappa} N(-\alpha\sigma\sqrt{u}) + \frac{1}{\kappa} \left( -\frac{1}{\sigma \sqrt{u}} \right) \int_0^\infty e^{-\kappa v} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{-v - \alpha\sigma^2 u}{\sigma \sqrt{u}} \right)^2 \right) \, dv
\]

Completing the squares on the second integral on the RHS, we get

\[
\int_0^\infty e^{-\kappa v} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{-v - \alpha\sigma^2 u}{\sigma \sqrt{u}} \right)^2 \right) \, dv = \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp \left( \frac{1}{2} \left( \frac{-v^2 - 2\alpha\sigma^2 u - \alpha^2\sigma^4 u^2}{\sigma^2 u} \right) \right) \left( -\frac{1}{2} \left( \frac{2\alpha\kappa\sigma^2 u}{\sigma^2 u} \right) \right) \, dv
\]

\[
= \exp \left( -\frac{1}{2} \left( \frac{\alpha^2 - (\alpha + \kappa)^2}{\sigma^2 u} \sigma^4 u^2 \right) \right) \times
\]

\[
\times \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{v + (\alpha + \kappa)\sigma^2 u}{\sigma \sqrt{u}} \right)^2 \right) \, dv
\]

\[
= \exp \left( \frac{1}{2} \left( (\alpha + \kappa)^2 - \alpha^2 \right) \sigma^2 u \right) \times
\]

\[
\times \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{-v - (\alpha + \kappa)\sigma^2 u}{\sigma \sqrt{u}} \right)^2 \right) \, dv
\]

This gives

\[
L \left[ e^{-\beta v} N \left( h(u,v) \right) \right] = \frac{1}{\kappa} N(-\alpha\sigma\sqrt{u}) + \frac{1}{\kappa} \exp \left( \frac{1}{2} \left( (\alpha + \kappa)^2 - \alpha^2 \right) \sigma^2 u \right)
\]

\[
\times \left( -\frac{1}{\sigma \sqrt{u}} \right) \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{-v - (\alpha + \kappa)\sigma^2 u}{\sigma \sqrt{u}} \right)^2 \right) \, dv
\]

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Changing variables, let \( w = \frac{-v-(\alpha+\kappa)\sigma^2u}{\sigma\sqrt{u}} \). Then, \( dw = -\frac{1}{\sigma\sqrt{u}} \, dv \), and

\[
v > 0 \iff w < \frac{-(\alpha + \kappa) \sigma^2 u}{\sigma \sqrt{u}} = - (\alpha + \kappa) \sigma \sqrt{u}
\]

and

\[
v < \infty \iff w > -\infty
\]

Therefore,

\[
- \frac{1}{\sigma \sqrt{u}} \, \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( -\frac{v - (\alpha + \kappa) \sigma^2 u}{\sigma \sqrt{u}} \right)^2 \right) \, dv = \int_{-\infty}^{-\frac{-(\alpha+\kappa)\sigma\sqrt{u}}{\sigma\sqrt{u}}} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} w^2 \right) \left( -\frac{1}{\sigma \sqrt{u}} \, dv \right) = N \left( - (\alpha + \kappa) \sigma \sqrt{u} \right)
\]

Plugging the expression above into the expression for \( L \left[ e^{-\beta v} N \left( h (u, v) \right) \right] \) and replacing \( \kappa \) by \( s + \beta \), we arrive at the result.

Another important result we will need ahead is the following

**Lemma 8: The Way Back: Inverting the Laplace Transforms of Bond Pricing Function Terms**

Define \( \Delta (v; x, w, p) \) as

\[
\Delta (v; x, w, p) \equiv L^{-1} \left\{ \frac{1}{s + p} \left\{ N \left( w \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s+x)^2 - w^2 \right] \sigma^2 m N \left( - (s+x) \sigma \sqrt{m} \right) } \right\} \right\}
\]

Then,

\[
\Delta (v; x, w, p) = \left( N \left( w \sigma \sqrt{m} \right) - e^{\frac{1}{2} \left[ (p-x)^2 - w^2 \right] \sigma^2 m N \left( (p-x) \sigma \sqrt{m} \right) } \right) e^{-pv} + e^{\frac{1}{2} \left[ (p-x)^2 - w^2 \right] \sigma^2 m v} e^{-pv} N \left( \frac{-v + (p-x) \sigma^2 m}{\sigma \sqrt{m}} \right)
\]

**Proof.** By equation D7,

\[
L \left[ e^{-pv} N \left( (p-x) \sigma \sqrt{m} \right) \right] = \frac{1}{s + p} N \left( (p-x) \sigma \sqrt{m} \right)
\]

Let \( \alpha = - (p-x) \), \( \beta = p \), so that \( \alpha + \beta = x \), claim D9 gives

\[
L \left[ e^{-pv} N \left( \frac{-(\alpha + \kappa) \sigma^2 u}{\sigma \sqrt{u}} \right) \right] = \frac{1}{s + p} \left\{ N \left( (p-x) \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s+x)^2 - (p-x)^2 \right] \sigma^2 m} \times N \left( - (s+x) \sigma \sqrt{m} \right) \right\}
\]

Combining the two equations above, we get

\[
L \left[ e^{-pv} N \left( \frac{-v + (p-x) \sigma^2 m}{\sigma \sqrt{m}} \right) \right] - L \left[ e^{-pv} N \left( (p-x) \sigma \sqrt{m} \right) \right] = \frac{1}{s + p} e^{\frac{1}{2} \left[ (s+x)^2 - (p-x)^2 \right] \sigma^2 m} N \left( - (s+x) \sigma \sqrt{m} \right)
\]
Multiplying both sides by $e^{\frac{1}{2}(p-x)^2\sigma^2m}$ and combining the terms on the LHS give
\[
L \left[ e^{\frac{1}{2}[(p-x)^2-w^2]\sigma^2m}e^{-pv} \left\{ -N \left( (p-x) \sigma \sqrt{m} \right) + N \left( \frac{-v + (p-x) \sigma^2m}{\sigma \sqrt{m}} \right) \right\} \right] = \frac{1}{s+p} e^{\frac{1}{2}[(s+x)^2-w^2]\sigma^2m}N \left( (s+x) \sigma \sqrt{m} \right)
\]

It follows that
\[
L^{-1} \left[ \frac{1}{s+p} e^{\frac{1}{2}[(s+x)^2-w^2]\sigma^2m}N \left( -(s+x) \sigma \sqrt{m} \right) \right] = -e^{\frac{1}{2}[(p-x)^2-w^2]\sigma^2m}e^{-pv} \times \left\{ N \left( (p-x) \sigma \sqrt{m} \right) - N \left( \frac{-v + (p-x) \sigma^2m}{\sigma \sqrt{m}} \right) \right\}
\]

We can now solve for $\Delta (v; x, w, p)$:
\[
\Delta (v; x, w, p) = L^{-1} \left[ \frac{1}{s+p} e^{\frac{1}{2}[(s+x)^2-w^2]\sigma^2m}N \left( -(s+x) \sigma \sqrt{m} \right) \right] + L^{-1} \left[ \frac{1}{s+p} N \left( w\sigma \sqrt{m} \right) \right]
\]
\[
= -e^{\frac{1}{2}[(p-x)^2-w^2]\sigma^2m}e^{-pv} \left\{ N \left( (p-x) \sigma \sqrt{m} \right) - N \left( \frac{-v + (p-x) \sigma^2m}{\sigma \sqrt{m}} \right) \right\} + N \left( w\sigma \sqrt{m} \right) e^{-pv}
\]

The result follows from rearranging the terms on the RHS. \qed

**Appendix E. Defaultable Bond Valuation**

This section derives the bond pricing formulas for the constant and stochastic volatility cases, by taking the pre- and post-volatility-shock bankruptcy barriers as given. Define the vector $\theta \equiv (r_{grow}, r_{disc}, \sigma)$ of interest rates and firm volatility. The value of a bond with maturity and cash-flows given by $b$, when the fundamental value of the firm is $V_t$ and the firm’s default barrier is $V_B$, is $d_{\sigma} (V_t, \tau; V_B, \omega, \theta)$, where subscript $\sigma$ indicates that volatility is fixed. By Ito’s Lemma, this pricing function must satisfy the following partial differential equation\(^\text{22}\):

\[
\begin{align*}
  r_{disc} \cdot d_{\sigma} (V_t, \tau; V_B, \omega, \theta) &= c + \frac{\partial}{\partial t} d_{\sigma} (V_t, \tau; V_B, \omega, \theta) + r_{grow} \cdot V_t \frac{\partial}{\partial V} d_{\sigma} (V_t, \tau; V_B, \omega, \theta) \\
  &\quad + \frac{1}{2} \sigma^2 V_t^2 \frac{\partial^2}{\partial V^2} d_{\sigma} (V_t, \tau; V_B, \omega, \theta)
\end{align*}
\]
(\text{E1})

with $r_{grow} = r - \bar{\sigma}$ and $r_{disc} = r + \xi \kappa$. A direct application of the Feynman–Kac Theorem to the PDE above shows that the investors’ de facto discount rate, $r_{disc}$, equals the sum of the risk-free interest rate, $r$, and a strictly positive liquidity risk premium term, $\xi \kappa$. This premium arises from the costly forced liquidation discussed above.\(^\text{23}\)

The bond pays the annual coupon up until maturity or default, whichever comes sooner. When maturity happens before default, a bond holder receives the principal value:

\[
d_{\sigma} (V_t, 0; V_B, \omega, \theta) = p, \quad \text{for all } V_t > V_B
\]
(E2)

Finally, to pin down the price of the bond, it remains to show investors’ cash-flows in case of bankruptcy. All outstanding debt comes due at default and debt claims have priority over equity

\(^{\text{22}}\)A derivation can be found in the online appendix, available here: https://abcarvalho.github.io/bond-liquidity/Online-Appendix/

\(^{\text{23}}\)In the Online Appendix A, I offer a more direct, and perhaps more intuitive, derivation of $r_{disc}$ in a more general setting that allows for arbitrary cash-flows.
shares ("absolute priority" rule). To repay investors, the firm’s assets are liquidated at a fractional cost \((1 - \alpha)\), \(\alpha \in (0, 1)\). The recovery value on the outstanding debt is thus proportional to the firm’s fundamental value and equals \(\alpha V_B\). For simplicity, I impose the pari-passu clause, under which all bonds have the same seniority, regardless of their time-to-maturity. Taken together, these rules imply that each investor receives an equal share of the firm’s liquidation value upon default

\[
d_{\tau} (V_B, \tau; V_B, \omega, \theta) = \frac{\alpha V_B}{\mu_b \cdot m}, \text{ for all } \tau \in [0, m]
\]

Boundary conditions E2 and E3 allow us to back out the value of the bond. Instead of solving the pricing PDE directly, however, I rely on the martingale approach and derive the present value of the bond’s expected cash-flows.

THEOREM 1: The value of a defaultable bond with time-to-maturity \(\tau\), coupon \(c\), and principal \(p\), when the volatility of the firm’s fundamental value is constant and equal to \(\sigma\), and the default boundary is \(V_B > 0\), is:

\[
d_{\tau} (V_t, \tau; V^B, \omega, \theta) = \frac{c}{r_{\text{disc}}} + e^{-r_{\text{disc}} \tau} \left[ p - \frac{c}{r_{\text{disc}}} \right] (1 - F(\tau, v_t; \theta_0)) + \left[ \frac{\alpha V_B}{\mu_b \cdot m} - \frac{c}{r_{\text{disc}}} \right] G(\tau, v_t; \theta)
\]

The book value of a firm’s debt is

\[
D^{crf} (r_{\text{disc}}, \omega) = \int_0^m b^{crf} (\tau; r_{\text{disc}}, \omega) \cdot \mu_b \cdot d\tau
\]

where \(b^{crf} (\tau; r_{\text{disc}}, \omega)\) is the book value of a bond with maturity \(\tau\), when the investor’s rate of discount is \(r_{\text{disc}}\) and the bond’s maturity, coupon and principal satisfy the capital structure \(\omega\):

\[
b^{crf} (\tau; r_{\text{disc}}, \omega) = \int_t^{\tau+t} e^{-r_{\text{disc}} (s-t)} c ds + e^{-r_{\text{disc}} \tau} \left[ p - \frac{c}{r_{\text{disc}}} \right]
\]

and superscript \(crf\) stands for “credit-risk-free”.

The recovery value on the firm’s debt is the minimum of the liquidation value of the firm, \(\alpha V_B\), and the book value of debt.

\[
Rec(\tau) = \min \left( \alpha V_B, D^{crf} (r_{\text{disc}}, \omega) \right)
\]

Suppose, by way of contradiction, that the liquidation value of the firm is greater than the amount it owes creditors. Since \(b^{crf} (\tau; r_{\text{disc}}, \omega) \geq d_{\tau} (V^B, \tau; V^B, \omega, \theta)\) for all \(\tau \in [0, m]\) and \(V_t \geq 0\), the book value must be greater than the market value of the firm’s debt. Therefore,

\[
\alpha V_B > D^{crf} (r_{\text{disc}}, \omega) \geq \int_0^m d_{\tau} (V, \tau; V^B, \omega, \theta) \cdot \mu_b \cdot d\tau, \quad V \geq V^B
\]

which can only happen if the firm is declared bankrupt when the value of equity is strictly positive, a contradiction!
where \( \omega \equiv (\mu_b, m, c, p), \theta_0 \equiv (r_{grow}, \sigma), \theta \equiv (r_{disc}, r_{grow}, \sigma), \)

\[
F (u, v_t; \theta_0) = N \left( h_1 (u, v_t; \theta_0) \right) + e^{-2\hat{\varphi}_u} N \left( h_2 (u, v_t; \theta_0) \right)
\]

\[
G (u, v_t; \theta) = e^{\left\{ \hat{\varphi}_u \hat{\varphi}_v \right\} v_t N \left( q_1 (u, v_t; \theta) \right) + e^{-\left\{ \hat{\varphi}_v \right\} v_t N \left( q_2 (u, v_t; \theta) \right)}
\]

\[
h_1 (u, v_t; \theta_0) = \frac{(-v_t - \hat{a} (\theta_0) \sigma^2 u)}{\sigma \sqrt{u}}, \quad h_2 (u, v_t; \theta_0) = \frac{(-v_t + \hat{a} (\theta_0) \sigma^2 u)}{\sigma \sqrt{u}}
\]

\[
q_1 (u, v_t; \theta) = \frac{(-v_t - \hat{\varphi}_u \hat{\varphi}_v \sigma^2 u)}{\sigma \sqrt{u}}, \quad q_2 (u, v_t; \theta) = \frac{(-v_t + \hat{\varphi}_u \hat{\varphi}_v \sigma^2 u)}{\sigma \sqrt{u}}
\]

\[
v_t \equiv \ln \left( \frac{V_t}{V_B} \right), \quad \hat{a} (\theta_0) \equiv \frac{r_{grow} - \frac{1}{2} \sigma^2}{\sigma^2}, \quad \hat{\varphi} (\theta) \equiv \left[ \frac{\hat{\varphi} (\theta_0)^2 \sigma^4 + 2 r_{disc} \sigma^2}{\sigma^2} \right]^{1/2}
\]

and \( N (\cdot) \) is the cumulative standard normal distribution.

**Proof.** See the bond pricing formula for the stochastic volatility setting, of which the constant volatility model is a special case. The result above follows from setting the volatility shock hazard rate to zero in the function in Theorem 2 below, and is thus omitted here. \( \square \)

**THEOREM 2:** The value of a defaultable bond with time-to-maturity \( \tau \), coupon \( c \) and principal \( p \), when the initial asset volatility is \( \sigma_1 \), with \( \sigma_1 < \sigma_h \), the pre-volatility shock default boundary is \( V_{t}^{B} \) and the post-volatility-shock default boundary is set arbitrarily to \( V_{h}^{B} \) is:

\[
d\lambda (V_t, \tau; V_t^{B}, V_h^{B}, \lambda, \omega, \theta) = \frac{c}{r_{disc} + \lambda} + e^{-(r_{disc} + \lambda)\tau} \left[ p - \frac{c}{r_{disc} + \lambda} \right] \left[ 1 - F \left( \tau, \ln \left( \frac{V_t}{V_t^{B}} \right); \sigma_1 \right) \right]
\]

\[
+ \left[ \frac{\alpha V_B}{\mu_b \cdot m} - \frac{c}{r_{disc} + \lambda} \right] G \left( \tau, \ln \left( \frac{V_t}{V_t^{B}} \right); \theta_1 \right)
\]

\[
+ \lambda \int_{t}^{t+\tau} e^{-(r_{disc} + \lambda)(s-t)} \left\{ \int_{V_t^{B}}^{\infty} d\varphi (V, \tau - (s-t); V_h^{B}, \omega, \theta_h) \times
\]

\[
\times \left[ -\frac{\partial}{\partial V} \Psi^{\varepsilon,\ell} \left( s-t, \ln \left( \frac{V}{V_t^{B}} \right); \ln \left( \frac{V_t}{V_t^{B}} \right), \sigma_1 \right) \right] dV \right\} \cdot ds \quad (E5)
\]

for \( V_t^{B} \in (0, V_0) \), where \( \theta \equiv (r_{grow}, r_{disc}, \sigma_1, \sigma_h), \theta_j \equiv (r_{disc}, r_{grow}, \sigma_j), j = l, h, r_{disc} \equiv r + \xi_k, r_{grow} \equiv r - \tilde{\delta}, \Psi^{\varepsilon,\ell} (u, \varpi; v, \sigma) \equiv N \left( \frac{-\varpi + v + \hat{a} (\sigma) \sigma^2 u}{\sigma \sqrt{u}} \right) - e^{-2\hat{\varphi}_u} N \left( \frac{-\varpi - v + \hat{\varphi}_u (\sigma)^2 u}{\sigma \sqrt{u}} \right) \)

and \( N (\cdot) \) is the cumulative standard normal distribution and the functions \( \hat{a}, \hat{\varphi}, F, G \) and \( d\varphi (\cdot; \cdot) \) are as defined in Theorem 1.

I derive first the pre-volatility-shock bond price above by taking the constant-volatility bond pricing function \( d\varphi (\cdot) \) formula as given. I then show that this function is a special case of the stochastic volatility formula.

**Proof.** [Proof of Theorems 1 and 2]

Let \( V_t^{B} \) and \( V_h^{B} \) denote the firm’s choices of pre- and post-volatility shock bankruptcy barriers, respectively. By assumption, absent risk management, an increase in the firm’s asset volatility can only happen once in the life of the firm. From the time the shock arrives, the firm’s
volatility regime is no longer random, so cash-flows can be priced by the constant-volatility formula $d_\sigma (V^\sigma, \tau - t^\sigma; V^B_h, \omega, \theta_h)$.

Knowledge of a bond’s valuation under constant volatility allows us to determine the pre-volatility-shock bond price by considering tree mutually exclusive events: (i) maturity before default and volatility shock ($t + \tau < t^d \land t^\sigma$), (ii) default prior to maturity and volatility shock ($t^d < t^\sigma \land (t + \tau)$), and (iii) volatility shock before maturity and default ($t^\sigma < t^d \land (t + \tau)$).

As before, let $\theta_{0,i} = (r\text{\_grow}, \sigma_i)$ and $\theta_i = (r\text{\_disc}, r\text{\_grow}, \sigma_i)$ for $i \in \{l, h\}$. Denoting by $CF_j$ the present value of the cash-flows in case of event $j$, we have

$$CF_j = \int_t^{t+\tau} e^{-r\text{\_disc}(s-t)} c \cdot ds + e^{-r\text{\_disc}\cdot \sigma} p$$

when $t + \tau < t^d \land t^\sigma$,

$$CF_{ii} = \int_t^{t^d} e^{-r\text{\_disc}(s-t)} c \cdot ds + e^{-r\text{\_disc}\cdot t^d} \frac{\alpha V^B}{\mu_b \cdot m}$$

when $t^d < t^\sigma \land (t + \tau)$, and, finally,

$$CF_{iii} = \int_t^{t^\sigma} e^{-r\text{\_disc}(s-t)} c \cdot ds + e^{-r\text{\_disc}(t^\sigma-t)} 1_{\{t^\sigma < t^d \land (t+\tau)\}} d_\sigma (V^\sigma, \tau - (t^\sigma - t); V^B_h, \omega, \theta_h)$$

when $t^\sigma < t^d \land (t + \tau)$.

Combining the three expressions above, we obtain

$$CF = \int_t^{t^\sigma \land t^d \land (t+\tau)} e^{-r\text{\_disc}(s-t)} c \cdot ds + 1_{\{t+\tau < t^\sigma \land t^d\}} \cdot e^{-r\text{\_disc}\cdot \sigma} p + 1_{\{t^d < t^\sigma \land (t+\tau)\}} e^{-r\text{\_disc}\cdot t^d} \frac{\alpha V^B}{\mu_b \cdot m}$$

$$+ e^{-r\text{\_disc}(t^\sigma-t)} 1_{\{t^\sigma < t^d \land (t+\tau)\}} d_\sigma (V^\sigma, \tau - (t^\sigma - t); V^B_h, \omega, \theta_h)$$

(E6)

The value of the bond prior to the volatility shock is then obtained via the martingale approach by computing the expectation of the cash-flow expression in E6. Denote this price function by $d_\lambda (V_t, \tau; V^B_l, V^B_h, \lambda, \omega, \theta_l)$, where subscript $\lambda$ serves to differentiate the function name from that of the baseline case. We have

$$d_\lambda (V_t, \tau; V^B_l, V^B_h, \lambda, \omega, \theta_l) = E^Q \left[ \int_t^{t^\lambda \land t^d \land (t+\tau)} e^{-r\text{\_disc}(s-t)} c \cdot ds + e^{-r\text{\_disc}\cdot \sigma} p 1_{\{t+\tau < t^\sigma \land t^d\}} ight.$$

$$+ e^{-r\text{\_disc}(t^\sigma-t)} 1_{\{t^\sigma < t^d \land (t+\tau)\}} d_\sigma (V^\sigma, \tau - (t^\sigma - t); V^B_h, \omega, \theta_h)$$

$$+ e^{-r\text{\_disc}(t^\lambda-t)} \left( \frac{\alpha V^B}{\mu_b \cdot m} \right) 1_{\{t^\lambda < t^\sigma \land (t+\tau)\}} \left| \mathcal{F}_t \right) \right]$$

(E7)

The RHS of equation E7 can be broken up into into 4 terms: (i) expected coupon payments before default and volatility shock, (ii) expected recovery value in case of default prior to volatility shock, (iii) present value of principal when maturity happens before default and volatility shock,
and (iv) expected payment when volatility shock happens before default and maturity.

\[
d_{\lambda} (V_t, \tau; V^B_t, V^B_h, \lambda, \omega, \theta_t) = E^Q \left[ \int_t^{t_\alpha \wedge d \wedge (t+\tau)} e^{-r_{\text{disc}}(s-t)} c \cdot ds \bigg| \mathcal{F}_t \right] \\
+ E^Q \left[ e^{-r_{\text{disc}}(t_\alpha-t)} \left( \frac{\alpha V^B_t}{\mu_h \cdot m} \right) 1 \{ t_\alpha < t_\alpha \wedge (t+\tau) \} \bigg| \mathcal{F}_t \right] \\
+ E^Q \left[ e^{-r_{\text{disc}} \tau} p 1 \{ t+\tau < t_\alpha \wedge t_\alpha \} \bigg| \mathcal{F}_t \right] \\
+ E^Q \left[ e^{-r_{\text{disc}}(t_\alpha-t)} 1 \{ t^\sigma < t_\alpha \wedge (t+\tau) \} d\sigma (V_t^\sigma, \tau - (t^\sigma - t) ; V^B_h, \omega, \theta_h) \bigg| \mathcal{F}_t \right] \\
\] (E8)

Using the formulas in Appendix D, I derive the value of the bond by analyzing each term on the RHS of equation E8 separately. The first term corresponds to the value of the bonds’ coupon strips prior to default, maturity and the volatility shock.

\[
E^Q \left[ \int_t^{t_\alpha \wedge d \wedge (t+\tau)} e^{-(\rho+\xi)(s-t)} c \cdot ds \bigg| \mathcal{F}_t \right] = c \cdot t^{t+\tau} e^{-r_{\text{disc}}(s-t)} E^Q \left[ 1 \{ t_\alpha \wedge d > s \} \bigg| \mathcal{F}_t \right] ds
\]

A direct application of Lemma 6 gives

\[
E^Q \left[ \int_t^{t_\alpha \wedge d \wedge (t+\tau)} e^{-r_{\text{disc}}(s-t)} c \cdot ds \bigg| \mathcal{F}_t \right] = c \cdot p_{\text{strips}} (V_t, \tau; V^B_t, \theta_t)
\]

\[
= \frac{c}{r_{\text{disc}} + \lambda} \left[ 1 - e^{-(r_{\text{disc}} + \lambda) \tau} \psi \left( \tau, \ln \left( \frac{V_t}{V^B_t} \right) ; \theta_0, \lambda \right) \right] - \frac{c}{r_{\text{disc}} + \lambda} G \left( \tau, \ln \left( \frac{V_t}{V^B_t} \right) ; (r_{\text{disc}} + \lambda, r_{\text{grow}}, \sigma_t) \right) \quad \text{(E9)}
\]

Because the default barrier is non-stochastic, the payoff value drops out of the expectation on the second term. By Lemma 5, this expression becomes

\[
E^Q \left[ e^{-r_{\text{disc}}(t_\alpha-t)} \left( \frac{\alpha V^B_t}{\mu_h \cdot m} \right) 1 \{ t_\alpha < t_\alpha \wedge (t+\tau) \} \bigg| \mathcal{F}_t \right] = \frac{\alpha V^B_t}{\mu_h \cdot m} \cdot p_{\text{default}} (V_t, \tau; V^B_t (\sigma_t), \theta_t)
\]

\[
= \frac{\alpha V^B_t}{\mu_h \cdot m} \cdot G \left( \tau, \ln \left( \frac{V_t}{V^B_t} \right) ; (r_{\text{disc}} + \lambda, r_{\text{grow}}, \sigma_t) \right) \quad \text{(E10)}
\]

Next, the present value of the bond’s principal is the discounted principal times the probability that maturity happens before default and before the arrival of a risky investment opportunity. This probability term corresponds to the price of the zero-coupon bond derived in Lemma 4.

\[
E^Q \left[ e^{-r_{\text{disc}} \tau} p 1 \{ t+\tau < t_\alpha \wedge t_\alpha \} \bigg| \mathcal{F}_t \right] = p \cdot p_{\text{zc}} (V_t, \tau; V^B_t, \theta_t)
\]

\[
= e^{-r_{\text{disc}} + \lambda) \tau} P_{\text{zc}} \left( \tau, \ln \left( \frac{V_t}{V^B_t} \right) ; \theta_0, \lambda \right) \quad \text{(E11)}
\]
At last, the formula for the value of the forth and last term in equation E8 was given in Lemma 7 and equals

$$1 \{ t^d \wedge t^z > t \} \frac{\lambda}{\tau} \int_t^{t+\tau} e^{-(r_{disc} + \lambda)(s-t)} \left[ \int \int \ldots \right] d\sigma \left(V, \tau - (s-t); V^B_h, \omega, \theta_h, \right) \times$$

$$\left[ - \frac{\partial}{\partial V} \Psi^{v,t} (s-t, \ln \left( \frac{V}{V^B_l}, \ln \left( \frac{V}{V^B_l}, \theta_0, \right) \right) \right] \cdot ds$$

(E12)

The pre-volatility-shock bond pricing function follows from plugging the formulas in equations E9 to E12 in to equation E8 and setting \( r_{disc} \) to \( r + \xi k \) per Lemma 1. Finally, the constant volatility bond pricing formula \( d_\lambda \left(V_t, \tau; V^B_l, \omega, \theta \right) \) in Theorem 1 assumes that \( t^o \rightarrow \infty \), which is achieved by setting \( \lambda \) to 0 in equation E5, that is,

$$d_\lambda \left(V_t, \tau; V^B_l, 0, \omega, \theta \right) = d_\sigma \left(V_t, \tau; V^B_l, \omega, \theta \right), \quad \forall V^B_l, V^B_h \geq 0,$$

\( \square \)

Appendix F. The Value of Equity when Volatility is Constant

This section follows closely the appendix in He and Xiong (2012), but with a few adjustments. I change the notation to make explicit the dependency of the bankruptcy barrier function on the measure of outstanding bonds, \( \mu_b \). I also allow for the possibility of arbitrary risk-management costs. These costs are paid continuously and take the form of foregone dividends, captured by the parameter \( \iota \) below. Accordingly, I distinguish between gross-dividends, \( \delta \), and net dividends, \( \delta \equiv \delta - \iota \).

I prove the following theorem and derive the value of the optimal default barrier, \( V^B_{\omega} (t, \omega, \theta) \). For ease of exposition, the derivations are divided into several steps and organized in subsections.

THEOREM 3: Define \( v \equiv \ln \left( \frac{V}{V^B} \right) \). The value of equity when volatility is constant is given by

$$E^\sigma (v; V^B, t, \omega, \theta) = \left( \frac{\delta - \iota}{r - r_{grow}} \right) V^B e^v - \frac{\delta - \iota}{\sigma^2 z \left( \theta_0, \right)} V^B e^{-\gamma \left( \theta_0, \right) v}$$

$$- \frac{(1 - \pi) C + (1 - e^{-r_{disc} m}) \left( p - c \frac{r_{disc}}{r_{disc}} \right)}{\sigma^2 z \left( \theta_0, \right)} \left[ \frac{1}{\eta \left( \theta_0, \right)} + \frac{1 - e^{-\gamma \left( \theta_0, \right) v}}{\gamma \left( \theta_0, \right)} \right]$$

$$+ \frac{1}{\sigma^2 z \left( \theta_0, \right)} e^{-r_{disc} m} \left( p - c \frac{r_{disc}}{r_{disc}} \right) A(v; \hat{a} \left( \theta_0, \right), \theta_0) + \left( \frac{\alpha V^B}{m} - c \frac{r_{disc}}{r_{disc}} \right) A(v; -\hat{z} \left( \theta, \right), \theta_0)$$

(F1)

\( ^{25} \)In the paper, the risk-management parameter is set to zero, so gross- and net-dividends coincide.
where \( r_{grow} \equiv r - \delta \), \( r_{disc} \equiv r + \xi \kappa \), \( \theta_0 \equiv (r_{grow}, \sigma) \), \( \theta \equiv (r_{disc}, r_{grow}, \sigma) \), and

\[
A(v; y, \theta_0) \equiv \frac{1}{z(\theta_0) - y} \left[ K(v; \hat{a}(\theta_0), y, \gamma(\theta_0)) + k(v; \hat{a}(\theta_0), -y, -\eta(\theta_0)) \right] + \frac{1}{z(\theta_0) + y} \left[ K(v; \hat{a}(\theta_0), -y, \gamma(\theta_0)) + k(v; \hat{a}(\theta_0), y, -\eta(\theta_0)) \right]
\]

\[
K(v; x, w, p) \equiv \left[ \frac{N(w \sigma \sqrt{m}) - e^{\frac{1}{2}[(p-x)^2 - w^2]} \sigma^2 m N((p-x) \sigma \sqrt{m})}{\sigma \sqrt{m}} \right] e^{-pv} + e^{\frac{1}{2}[(p-x)^2 - w^2]} \sigma^2 m e^{-pv} N \left( \frac{-v + (p-x) \sigma^2 m}{\sigma \sqrt{m}} \right)
\]

\[
k(v; x, w, p) \equiv K(v; x, w, p) - e^{\eta \hat{V}(v; x, w, p)}
\]

\[
\eta(\theta) = z(\theta_0) - \hat{a}(\theta_0) > 1, \quad \gamma(\theta_0) = z(\theta_0) + \hat{a}(\theta_0)
\]

\[
\hat{a}(\theta_0) = \frac{r_{grow} - \frac{1}{2} \sigma^2}{\sigma^2}, \quad z(\theta_0) = \left( \frac{\hat{a}(\theta_0)^2 \sigma^4 + 2 \sigma^2 r}{\sigma^2} \right)^{1/2}, \quad \hat{z}(\theta) = \left( \frac{\hat{a}(\theta_0)^2 \sigma^4 + 2 \sigma^2 r_{disc}}{\sigma^2} \right)^{1/2}
\]

and \( K^{[n]} \) is the sum of the coefficients of \( e^{\eta(\theta_0) v} \) in \( K \).

**F.1 The Equity Value Partial Differential Equation**

The equity value \( E_\pi(V_t; t, \omega, \theta) \) satisfies

\[
r E_\pi^\pi(V_t; t, \omega, \theta) = r_{grow} V_t E_\pi^\pi(V_t; t, \omega, \theta) + \frac{1}{2} \sigma^2 V_t^2 E_\pi^\pi \left( \frac{\delta - \pi}{V_t} \right) - (1 - \pi) \left( \mu_b \cdot c \cdot m \right) + \mu_b \left[ d_\pi(V_t; m; V_B, \omega, \theta) - p \right] \tag{F2}
\]

**Proof.** At every instant \( dt \), a fraction \( dt \) of the firm’s debt matures and must be rolled over at the cost of \( \mu_b \left[ d_\pi(V_t; m; V_B, \omega, \theta) - p \right] \). In addition, the firm generates net dividends \( (\delta - \pi) V_t \) and must pay coupon \( (1 - \pi) C \). The instantaneous net cash-flow is then (omitting \( dt \)):

\[
\varphi(V_t; t, \omega, \theta) = (\delta - \pi) V_t - \mu_b \left[ (1 - \pi) (c \cdot m) + d_\pi(V_t; m; V_B, \omega, \theta) - p \right]
\]

The equity value Bellman equation is

\[
E_\pi^\pi(V_t; t, \omega, \theta) = \lim_{dt \downarrow 0} E_\pi^Q \left[ \varphi(V_t; m, t, \omega, \theta) dt + e^{-r dt} E_\pi^Q \left[ E_\pi^\pi(V_{t+dt}; t, \omega, \theta) \bigg| \mathcal{F}_{t+dt} \right] \bigg| \mathcal{F}_t \right]
\]

Approximating \( e^{rdt} \) by \( 1 + r dt \) and rearranging terms gives

\[
0 = \lim_{dt \downarrow 0} \left\{ E_\pi^Q \left[ \varphi(V_t; t, \omega, \theta) (1 + r dt) dt + E_\pi^Q \left[ E_\pi^\pi(V_{t+dt}; t, \omega, \theta) - E_\pi^\pi(V_t; t, \omega, \theta) \bigg| \mathcal{F}_{t+dt} \right] \bigg| \mathcal{F}_t \right] \right\}
\]

Dividing by \( dt \) and taking limits yield

\[
0 = \varphi(V_t; t, \omega, \theta) + \lim_{dt \downarrow 0} \frac{1}{dt} : E_\pi^Q \left[ E_\pi^\pi(V_{t+dt}; t, \omega, \theta) - E_\pi^\pi(V_t; t, \omega, \theta) \bigg| \mathcal{F}_t \right] - r E_\pi^\pi(V_t; t, \omega, \theta)
\]
By Ito’s Lemma,
\[
\lim_{dt \downarrow 0} \frac{1}{dt} \mathbb{E}^Q \left[ E^Q (V_{t+dt}; \iota, \omega, \theta) - E^Q (V_t; \iota, \omega, \theta) \right| \mathcal{F}_t] = r_{\text{grow}} V_t E^Q (V_t; \iota, \omega, \theta) + \frac{1}{2} \sigma^2 V_t^2 E_{VV}^Q (V_t; \iota, \omega, \theta)
\]

By combining the last two equations, we arrive at the solution. \(\square\)

Define
\[
v_t \equiv \ln \left( \frac{V_t}{V_B} \right)
\]

Rewriting the pricing PDE in terms of \(v_t\), we get
\[
r E^Q (v_t; \iota, \omega, \theta) = \left( \frac{\delta - \iota}{1 - \iota} \right) V_B e^{v_t} + \left( r_{\text{grow}} - \frac{1}{2} \sigma^2 \right) E^Q (v_t; \iota, \omega, \theta) + \frac{1}{2} \sigma^2 E_{vv}^Q (v_t; \iota, \omega, \theta)
\]
\[
+ \mu_b \left[ -(1 - \pi) (c \cdot m) + d \sigma (V_t, m; V_B, \omega, \theta) - p \right]
\]

Proof. Start by noticing that \(V_t = V_B e^{v_t}\) and \(\partial v_t / \partial V = 1 / V_t\). Next, consider the first and second derivatives of \(E (\cdot)\) with respect to \(v\):
\[
E^Q (v_t; \iota, \omega, \theta) = E^Q (V_t; \iota, \omega, \theta) V_t
\]
and
\[
E_{vv}^Q (v_t; \iota, \omega, \theta) = E_{VV}^Q (V_t; \iota, \omega, \theta) (V_t)^2 + E^Q (V_t; \iota, \omega, \theta) \frac{\partial V_t}{\partial v_t}
\]
\[
= V_t^2 E_{VV}^Q (V_t; \iota, \omega, \theta) + E^Q (v_t; \iota, \omega, \theta)
\]

Solving for \(E_{VV}^Q (\cdot)\) in terms of \(v\) yields
\[
E_{VV}^Q (V_t; \iota, \omega, \theta) = \frac{1}{V_t^2} \left[ E_{vv}^Q (v_t; \iota, \omega, \theta) - E^Q (v_t; \iota, \omega, \theta) \right]
\]

Replacing \(E^Q\) and \(E_{VV}^Q\) in the equity value PDE F2 by their expression in equations F4 and F5 above gives the result. \(\square\)

\textbf{F.2 Boundary Conditions}

To solve the PDE and back out the equity function, I impose two boundary conditions and require differentiability of the function at the bankruptcy barrier. The first condition is known as the limited liability of equity. In the event of default, bond investors have a claim to the residual value of the firm, but cannot seize any other assets from equity holders. Therefore, management chooses to default optimally when the value of equity falls to zero. At the bankruptcy boundary, we must have
\[
E^Q (V_B; \iota, \omega, \theta) = 0 \quad \text{(limited liability)}
\]

At the other extreme, when \(V \to \infty\), the equity function must converge to a linear function. This limiting condition is established in the next two claims.

CLAIM 4: \textit{For large enough values of} \(V\), \textit{the credit risk vanishes, so the value of a bond approaches that of a credit-risk-free bond.}\(\textsuperscript{26}\)

\[
b_{\text{ref}} (\tau; r_{\text{disc}}, \omega) = \left( 1 - e^{-r_{\text{disc}} \tau} \right) \cdot c + e^{-r_{\text{disc}} \tau} \cdot p, \quad \tau \in [0, m]
\]

\(\textsuperscript{26}\)This value may be distinct from that of a purely risk-free bond, since the discount rate may incorporate adjustments for other sources of risk, in particular the liquidity risk faced by investors, in which case \(r_{\text{disc}} > \tau\).
Proof. The result follows from noticing that
\[ \lim_{V_t \to \infty} F \left( \tau, \ln \left( \frac{V_t}{V_B} \right); \theta \right) = 0 \]
and
\[ \lim_{V_t \to \infty} G \left( \tau, \ln \left( \frac{V_t}{V_B} \right); \theta \right) = 0 \]
in the formula for \( d(\tau, \omega; \theta) \) in Theorem 1.

CLAIM 5: The equity function converges to a linear function as the underlying value of assets \( V \) increases:
\[ \bar{E}(V; \tau, \omega, \theta) \to \left( \frac{\delta - \nu}{r - r_{\text{grow}}} \right) V + \mu_b - (1 - \pi) (c \cdot m) + b_{\text{crf}} (m; r_{\text{disc}}, \omega) - p \quad \text{as } V \to \infty \quad \text{(linearity)} \]

Notice that the limiting polynomial does not depend on \( \sigma \).

Proof. If \( E^\sigma \) is continuous in \( V \) in any closed interval of the form \([V, \bar{V}]\) for \( V \geq V^B \), as will be shown to be the case, by the Stone–Weierstrass theorem it can be approximated by a polynomial \( p(\cdot) \) in \([V, \bar{V}]\). I now prove that \( p(\cdot) \) is of degree one.

For very large \( V \), by claim 4, we can replace \( d(\tau, \omega; \theta) \) by \( b_{\text{crf}} (m; r_{\text{disc}}, \omega) \) in equation F2 to obtain
\[ rE^\sigma (V; \tau, \omega, \theta) = r_{\text{grow}} E_V^\sigma (V; \tau, \omega, \theta) + \frac{1}{2} \sigma^2 V^2 E^\sigma_V (V; \tau, \omega, \theta) \]
\[ + (\delta - \nu) V - (1 - \pi) \left( \mu_b \cdot c \cdot m \right) + \mu_b \left[ b_{\text{crf}} (m; r_{\text{disc}}, \omega) - p \right] \]

Let the limiting polynomial be
\[ p(V) = \sum_{j=0}^{n} j \phi_j V^{j-1}, \quad n > 0 \]

Replacing \( E^\sigma \) with \( p(\cdot) \), we get
\[ r \sum_{j=0}^{n} \phi_j V^j = r_{\text{grow}} \left[ \sum_{j=0}^{n} j \phi_j V^{j-1} \right] + \frac{1}{2} \sigma^2 V^2 \left[ \sum_{j=0}^{n} j (j - 1) \phi_j V^{j-2} \right] + (\delta - \nu) V \]
\[ + \left[ - (1 - \pi) C + \mu_b \left( b_{\text{crf}} (m) - p \right) \right] \equiv H \]

This gives
\[ r \sum_{j=0}^{n} \phi_j V^j = r_{\text{grow}} \left[ \sum_{j=1}^{n} j \phi_j V^{j-1} \right] + \frac{1}{2} \sigma^2 \left[ \sum_{j=2}^{n} j (j - 1) \phi_j V^{j-1} \right] + \delta V + H \]

Rearranging and collecting terms, we obtain
\[ [(r - r_{\text{grow}}) \phi_1 - (\delta - \nu)] V + \sum_{j=2}^{n} \phi_j \left\{ r - r_{\text{grow}} j - \frac{1}{2} \sigma^2 j (j - 1) \right\} V^j = H - r \phi_0 \]

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Since the equation above must be valid for all sufficiently large values of $V$, the coefficients of $p$ on all terms of degree higher than one must be zero

$$\phi_j = 0, \quad \forall j \geq 2$$

Therefore, as $d_\sigma (V, m; V^B, \omega, \theta) \to b^{\text{opt}} (m; r_{\text{disc}}, \omega)$, $E^\sigma (V; \iota, \omega, \theta)$ converges to $p (V) = \phi_1 (\iota) V + \phi_0$, with

$$\phi_0 = \frac{H}{r} = \mu_b - (1 - \pi) (c \cdot m) + b^{\text{opt}} (m; r_{\text{disc}}, \omega) - p$$

$$\phi_1 (\iota) = \frac{\delta - \iota}{r - r_{\text{grow}}}$$

where I make explicit the dependency of the $\phi_1$ term on the risk-management costs, $\iota$. Q.E.D. \qed

**COROLLARY 3:** When risk-management costs are zero, the equity function converges to the difference between the underlying value of assets $V$ and the present value of the coupon payments and rollover costs of a credit-risk-free debt:

$$V + \mu_b - (1 - \pi) (c \cdot m) + b^{\text{opt}} (m; r_{\text{disc}}, \omega) - p$$

Finally, I require the equity function to be smooth near the default barrier by imposing a smooth-pasting condition

$$\lim_{V \downarrow V^B} \frac{\partial}{\partial V} E^\sigma (V; \iota, \omega, \theta) = 0 \quad \text{(smooth-pasting)} \quad (F9)$$

In the calculations that follow, I begin by assuming that $E^\sigma (v; \iota, \omega, \theta)$ is differentiable at $v$, $E^\sigma_v (0; \iota, \omega, \theta) = l$ for some $l \in \mathbb{R}$, where subscript $v$ denotes the first derivative of the function with respect to $v$. Later on I invoke the condition in equation $F9$ and show that $l = 0$.

The linearity and smooth-pasting conditions will also help us pin down the value of the optimal default barrier, $V^B$.

### F.3 The Laplace Transform of the Equity Function when Volatility is Constant

Define the Laplace transformation of $E^\sigma (v; \iota, \omega, \theta)$ as

$$Q (s) \equiv L [E^\sigma (v; \iota, \omega, \theta)] = \int_0^\infty e^{-sv} E^\sigma (v; \iota, \omega, \theta) \, dv \quad (F10)$$

Applying the transformation to both sides of the PDE $F3$ gives

$$rL [E^\sigma (v; \iota, \omega, \theta)] = \left( r_{\text{grow}} - \frac{1}{2} \sigma^2 \right) L [E^\sigma_v] + \frac{1}{2} \sigma^2 L [E^\sigma_{vv}] + \mu_b L [d_\sigma (v, m; \omega, \theta)]$$

$$+ (\delta - \iota) V^B L [e^{\iota v}] - [(1 - \pi) C + \mu_b \cdot p] L [1]$$

where the constants come out of the operator $L$ by the linearity of the Laplace transformation.

By equation $D7$ in section $D.2$,

$L [1] = L [e^{0v}] = \frac{1}{s}$

$L [e^v] = \frac{1}{s - 1}$
so the transformation above becomes

\[
rQ(s) = \left( r_{\text{grow}} - \frac{1}{2}\sigma^2 \right) L \left[ E^\sigma_v \right] + \frac{1}{2}\sigma^2 L \left[ E^\sigma_{vv} \right] + \mu_b L \left[ d\sigma(v_t, m) \right] \\
+ \frac{(\delta - \iota) V_B}{s - 1} - \frac{(1 - \pi) C + \mu_b \cdot \mathbf{p}}{s}
\]

(F11)

CLAIM 6: The Laplace Transformation of the first and second derivatives of \( E^\sigma(v; \iota, \omega, \theta) \) are

\[
L \left[ E^\sigma_v(v; \iota, \omega, \theta) \right] = sQ(s)
\]  
(F12)

and

\[
L \left[ E^\sigma_{vv}(v; \iota, \omega, \theta) \right] = s^2Q(s) - l
\]

(F13)

for some \( l \in \mathbb{R} \).

Proof. Using integration by parts on the RHS of equation F10 yields

\[
Q(s) \equiv \int_0^\infty e^{-sv}E^\sigma(v; \iota, \omega, \theta) \, dv \\
= \left( -\frac{e^{-sv}}{s} E^\sigma_v(v; \iota, \omega, \theta) \right) \Big|_{v=0}^\infty - \int_0^\infty \left( -\frac{e^{-sv}}{s} \right) E^\sigma_{vv}(v; \iota, \omega, \theta) \, dv
\]

Rearranging terms and imposing the bankruptcy boundary condition gives

\[
L \left[ E^\sigma_v(v; \iota, \omega, \theta) \right] = sF(s) - \frac{E^\sigma_v(0; \iota, \omega, \theta)}{s} = sQ(s)
\]

Using integration by parts again, \( L \left[ E^\sigma_v(v; \iota, \omega, \theta) \right] \) can be written in terms of \( L \left[ E^\sigma_{vv}(v; \iota, \omega, \theta) \right] \) as follows

\[
L \left[ E^\sigma_v(v; \iota, \omega, \theta) \right] = \int_0^\infty e^{-sv}E^\sigma_v(v; \iota, \omega, \theta) \, dv \\
= \left( -\left( -\frac{e^{-sv}}{s} E^\sigma_v(v; \iota, \omega, \theta) \right) \Big|_{v=0}^\infty \right) - \int_0^\infty \left( -\frac{e^{-sv}}{s} \right) E^\sigma_{vv}(v; \iota, \omega, \theta) \, dv
\]

The limited-liability condition in equation F6 requires that \( E^\sigma(0; \iota, \omega, \theta) = 0 \). In addition, per the differentiability of the equity function at the bankruptcy barrier, we must have

\[
E^\sigma_v(0; \iota, \omega, \theta) = l
\]

for some \( l \in \mathbb{R} \), as discussed in section F.2. These two constraints give

\[
L \left[ E^\sigma_{vv}(v; \iota, \omega, \theta) \right] = s^2Q(s) - \frac{sE^\sigma_v(0; \iota, \omega, \theta)}{s} - \frac{E^\sigma_v(0; \iota, \omega, \theta)}{s} \cdot l
\]

Replacing \( L \left[ E^\sigma_v(v; \iota, \omega, \theta) \right] \) and \( L \left[ E^\sigma_{vv}(v; \iota, \omega, \theta) \right] \) in equation F11 by their formulas in F12 and F13 yields

\[
\left[ r - \left( r_{\text{grow}} - \frac{1}{2}\sigma^2 \right) s - \frac{1}{2}\sigma^2 s^2 \right] Q(s) = \mu_b L \left[ d\sigma(v_t, m; \omega, \theta) \right] - \frac{1}{2}\sigma^2 l + \frac{(\delta - \iota) V_B}{s - 1} - \frac{(1 - \pi) C + \mu_b \cdot \mathbf{p}}{s}
\]

(F14)
Now let’s analyze the following equation
\[-\frac{1}{2}\sigma^2s^2 - \left(r_{\text{grow}} - \frac{1}{2}\sigma^2\right)s + r = 0\]

Denote by \(\eta(\theta_0) > 0\) and \(-\gamma(\theta_0) < 0\) the two roots of the equation with respect to \(s\) (Notice \(\gamma(\theta_0)\) is the negative of one of the roots.). Then
\[-\frac{1}{2}\sigma^2[(s - \eta(\theta_0))(s + \gamma(\theta_0))] = 0\]

It is easy to see that
\(\eta(\theta_0) \equiv z(\theta_0) - \hat{a}(\theta_0) > 0\), \(\gamma(\theta_0) \equiv (\hat{a}(\theta_0) + z(\theta_0)) > 0\)

where
\(\hat{a}(\theta_0) \equiv \frac{(r_{\text{grow}} - \frac{1}{2}\sigma^2)}{\sigma^2}\)
\(z(\theta_0) \equiv \left[\frac{(\hat{a}(\theta_0)^2\sigma^4 + 2\sigma^2r)}{\sigma^2}\right]^{1/2}\)

and \(\theta_0 \equiv (r_{\text{grow}}, \sigma)\).

We shall make use of the following result to pin down the solution to \(E^\pi(v; \iota, \omega, \theta)\).

CLAIM 7: The positive root of \(-\frac{1}{2}\sigma^2s^2 - \left(r_{\text{grow}} - \frac{1}{2}\sigma^2\right)s + r = 0\) is greater than unity: \(\eta(\theta_0) > 1\).

Proof. Start by noticing that
\(\hat{a}(\theta_0) = \frac{r}{\sigma^2} - \left(\frac{r - r_{\text{grow}} + \sigma^2/2}{\sigma^2}\right) < \frac{r}{\sigma^2} - \frac{1}{2}\)

This gives
\(\frac{r}{\sigma^2} > \hat{a}(\theta_0) + \frac{1}{2}\)

Then
\(z(\theta_0) - \hat{a}(\theta_0) = \left[\frac{(\hat{a}(\theta_0)^2\sigma^2 + 2r\sigma^2}{\sigma^2}\right]^{1/2} - \hat{a}(\theta_0)\)
\(> \left[(\hat{a}(\theta_0) + 1)^2\right]^{1/2} - \hat{a}(\theta_0)\)

The result follows by noticing that
\(\eta(\theta_0) > |\hat{a}(\theta_0) + 1| - \hat{a}(\theta_0) = \begin{cases} 1 & \text{if } \hat{a}(\theta_0) \geq -1 \\ 1 - 2\hat{a}(\theta_0) & \text{if } \hat{a}(\theta_0) < -1 \end{cases}\)

This gives
\(\eta(\theta_0) \equiv z(\theta_0) - \hat{a}(\theta_0) > 1, \quad \gamma(\theta_0) \equiv \hat{a}(\theta_0) + z(\theta_0) > 0\) (F15)
By the partial fraction technique\textsuperscript{27},
\[
\frac{1}{s - \omega_1} \cdot \frac{1}{s - \omega_2} = \frac{1}{s - \omega_1} - \frac{1}{s - \omega_2}
\] (F16)

Therefore,
\[
\frac{1}{(s - \eta (\theta_0)) (s + \gamma (\theta_0))} = \frac{1}{s - \eta (\theta_0)} - \frac{1}{s + \gamma (\theta_0)}
\] (F17)

Using the results in F15 and F17, we can rewrite equation F14 as follows
\[
\frac{1}{2} \sigma^2 Q (s) = \frac{1}{(s - \eta (\theta_0)) - \frac{1}{(s + \gamma (\theta_0))}} \left\{ \mu_b L [d_\sigma(v_t,m)] - \frac{1}{2} \sigma^2 l + \frac{(\bar{\delta} - \mathcal{I}) V^B}{s - 1} - \frac{(1 - \pi) C + \mu_b \cdot p}{s} \right\}
\] (F18)

\textbf{F.4 \ Laplace Transform of The Bond Price Function when Volatility is Constant}

Now let’s consider the laplace transformation of the bond price function in the constant volatility case. The formula for \(d_\sigma(v_t,m)\) was given in equation 8. By the linearity of the \(L\) operator, we have
\[
L \left[ d_\sigma (v, m; V^B, \theta) \right] = \left\{ \frac{c}{r_{\text{disc}}} + e^{-r_{\text{disc}} m} \left( p - \frac{c}{r_{\text{disc}}} \right) \right\} L [1] - e^{-r_{\text{disc}} m} \left[ p - \frac{c}{r_{\text{disc}}} \right] L [F (m, v; \theta_0)] + \frac{\alpha V^B}{\mu_b \cdot m - \frac{c}{r_{\text{disc}}}} L [G (m, v; \theta)]
\]

Plugging this expression into equation F18, we obtain
\[
\frac{1}{2} \sigma^2 Q (s) = \frac{1}{(s - \eta (\theta_0)) - \frac{1}{(s + \gamma (\theta_0))}} \left\{ \frac{(\bar{\delta} - \mathcal{I}) V^B}{s - 1} - \frac{(1 - \pi) C + \mu_b \left( 1 - e^{-r_{\text{disc}} m} \right)}{s} \left( p - \frac{c}{r_{\text{disc}}} \right) \right\} - \frac{1}{2} \sigma^2 l
\] (F19)

Define
\[
\hat{Q} (s) \equiv - \frac{1}{(s - \eta (\theta_0)) - \frac{1}{(s + \gamma (\theta_0))}} \left\{ \frac{(\bar{\delta} - \mathcal{I}) V^B}{s - 1} - \frac{(1 - \pi) C + \mu_b \left( 1 - e^{-r_{\text{disc}} m} \right)}{s} \left( p - \frac{c}{r_{\text{disc}}} \right) \right\} - \frac{1}{2} \sigma^2 l
\] (F20)

\[
\overline{Q} (s) \equiv - \frac{1}{(s - \eta (\theta_0)) - \frac{1}{(s + \gamma (\theta_0))}} \left\{ - \mu_b \cdot e^{-r_{\text{disc}} m} \left[ p - \frac{c}{r_{\text{disc}}} \right] L [F (m, v; \theta_0)] + \mu_b \left[ \frac{\alpha V^B}{\mu_b \cdot m - \frac{c}{r_{\text{disc}}}} \right] L [G (m, v; \theta)] \right\}
\] (F21)

so that
\[
\frac{1}{2} \sigma^2 Q (s) = \hat{Q} (s) + \overline{Q} (s)
\]

We shall solve for these two expressions separately.

\textsuperscript{27} This result follows from solving \( \frac{1}{s - \omega_1} \cdot \frac{1}{s - \omega_2} = \frac{A}{s - \omega_1} \cdot \frac{B}{s - \omega_2} \) for \( A \) and \( B \).
By the partial fraction technique in equation F16,
\[
- \left( \frac{1}{(s - \eta(\theta_0))} - \frac{1}{(s + \gamma(\theta_0))} \right) \frac{1}{s - 1} = -\frac{1}{\eta(\theta_0)} - 1 \left[ \frac{1}{s - \eta(\theta_0)} - \frac{1}{s - 1} \right] - \frac{1}{\gamma(\theta_0) + 1} \left[ \frac{1}{s + \gamma(\theta_0)} - \frac{1}{s - 1} \right]
\]
and
\[
\frac{1}{\eta(\theta_0) + \gamma(\theta_0)} - \frac{1}{s - 1} \frac{1}{\eta(\theta_0) + \gamma(\theta_0)} \left\{ \frac{1}{\eta(\theta_0)} - \frac{1}{s - 1} - \frac{1}{\gamma(\theta_0) + 1} \left[ \frac{1}{\eta(\theta_0)} - \frac{1}{s - 1} \right] \right\}
\]
Substituting these expressions in equation F20 gives
\[
\hat{Q}(s) = -\left( \frac{\delta - \psi}{\eta(\theta_0) + \gamma(\theta_0)} \right) V_B \left\{ \frac{1}{s - \eta(\theta_0)} - 1 \left[ \frac{1}{s - \eta(\theta_0)} - \frac{1}{s - 1} \right] + \frac{1}{\gamma(\theta_0) + 1} \left[ \frac{1}{s + \gamma(\theta_0)} - \frac{1}{s - 1} \right] \right\}
\]
Define its laplace inverse as \( \hat{\mathcal{E}}(v; \iota, \omega, \theta) \equiv L^{-1} \left[ \hat{Q}(s) \right] \). Then
\[
\frac{2}{\sigma^2} \hat{\mathcal{E}}(v; \iota, \omega, \theta) = \left( \frac{\delta - \psi}{r - r_{grow}} \right) V_B e^v - \left( \frac{\delta - \psi}{\sigma^2 z(\theta_0)} \right) V_B \left[ \frac{e^{\eta(\theta_0)v}}{\eta(\theta_0) - 1} + \frac{e^{-\gamma(\theta_0)v}}{\gamma(\theta_0) + 1} \right] + \frac{l}{2z(\theta_0)} \left( e^{\eta(\theta_0)v} - e^{-\gamma(\theta_0)v} \right)
\]
\[
+ \frac{(1 - \pi) C + \mu_b (1 - e^{-\tau_{disc}})}{\sigma^2 z(\theta_0)} \left\{ \frac{1}{\eta(\theta_0)} \left( e^{\eta(\theta_0)v - 1} \right) - \frac{1}{\gamma(\theta_0)} \left( 1 - e^{-\gamma(\theta_0)v} \right) \right\}
\]
\[\text{Proof. Abusing notation slightly by omitting } \theta_0, \text{ we have}
\]
\[
\hat{\mathcal{E}}(v; \iota, \omega, \theta) = -\left( \frac{\delta - \psi}{\eta + \gamma} \right) V_B \left\{ \frac{1}{\eta - 1} \left( L^{-1} \left[ \frac{1}{s - \eta} \right] - L^{-1} \left[ \frac{1}{s - 1} \right] \right) + \frac{1}{\gamma + 1} \left( L^{-1} \left[ \frac{1}{s + \gamma} \right] - L^{-1} \left[ \frac{1}{s - 1} \right] \right) \right\}
\]
\[
+ \frac{(1 - \pi) C + \mu_b (1 - e^{-\tau_{disc}})}{\eta + \gamma} \left\{ \frac{1}{\eta} \left( L^{-1} \left[ \frac{1}{s - \eta} \right] - L^{-1} \left[ \frac{1}{s - 1} \right] \right) - \frac{1}{\gamma} \left( L^{-1} \left[ \frac{1}{s + \gamma} \right] - L^{-1} \left[ \frac{1}{s + \gamma} \right] \right) \right\}
\]
\[
+ \frac{1}{2} \frac{\sigma^2 L}{\eta + \gamma} \left( L^{-1} \left[ \frac{1}{s + \gamma} \right] - L^{-1} \left[ \frac{1}{s + \gamma} \right] \right)
\]
\[
= -\left( \frac{\delta - \psi}{\eta + \gamma} \right) V_B \left\{ \frac{1}{\eta - 1} \left( e^{\eta v} - e^v \right) + \frac{1}{\gamma + 1} \left( e^{-\gamma v} - e^v \right) \right\}
\]
\[
+ \frac{(1 - \pi) C + \mu_b (1 - e^{-\tau_{disc}})}{\eta + \gamma} \left\{ \frac{1}{\eta} \left( e^{\eta v} - 1 \right) - \frac{1}{\gamma} \left( 1 - e^{-\gamma v} \right) \right\}
\]
\[
+ \frac{1}{2} \frac{\sigma^2 L}{\eta + \gamma} \left( e^{\eta v} - e^{-\gamma v} \right)
\]
Notice now that
\[
\frac{(\delta - \iota) V_B}{\eta + \gamma} e^v \left\{ \frac{1}{\eta - 1} + \frac{1}{\gamma + 1} \right\} = \frac{(\delta - \iota) V_B}{(\eta - 1)(\gamma + 1)} e^v
\]
Moreover,
\[
(\eta - 1)(\gamma + 1) = z^2 - a^2 - 2a - 1 = 2 \left( r - r_{\text{grow}} \right)
\]
which gives
\[
\frac{(\delta - \iota) V_B}{\eta + \gamma} e^v \left\{ \frac{1}{\eta - 1} + \frac{1}{\gamma + 1} \right\} = \frac{\sigma^2}{2} \left( \frac{\delta - \iota}{r - r_{\text{grow}}} \right) V_B e^v
\]
Collecting terms and plugging the expression above in the formula for \( \hat{E}^\pi (v; \iota, \omega, \theta) \) yields
\[
\hat{E}^\pi (v; \iota, \omega, \theta) = \frac{\sigma^2}{2} \left( \frac{\delta - \iota}{r - r_{\text{grow}}} \right) V_B e^v - \frac{\delta - \iota}{\eta + \gamma} V_B \left\{ \frac{1}{\eta - 1} e^{\eta v} + \frac{1}{\gamma + 1} e^{-\gamma v} \right\}
\]
\[
+ \frac{(1 - \pi) C + \mu_b (1 - e^{-r_{\text{disc}}}) \left( \frac{p - e^{c \iota}}{r_{\text{disc}}} \right)}{\eta + \gamma} \left\{ \frac{1}{\eta} (e^{\eta v} - 1) - \frac{1}{\gamma} (1 - e^{-\gamma v}) \right\}
\]
\[
+ \frac{1}{2} \sigma^2 l \frac{1}{\eta + \gamma} \left( e^{\eta v} - e^{-\gamma v} \right)
\]
The result follows from substituting \( \eta + \gamma = 2z \) in the equation above.

F.6 The Inverse Laplace Transform of \( \overline{Q} (s) \)

To solve for \( \overline{Q} (s) \), define the auxiliary terms
\[
\Gamma_1 (s) \equiv \left( \frac{1}{(s - \eta (\theta_0))} - \frac{1}{(s + \gamma (\theta_0))} \right) L [F (m, v; (\theta_0))]
\]
and
\[
\Gamma_2 (s) \equiv \left( \frac{1}{(s - \eta (\theta_0))} - \frac{1}{(s + \gamma (\theta_0))} \right) L [G (m, v; \theta)]
\]
so that equation \text{F.21} can be rewritten as
\[
(\eta (\theta_0) + \gamma (\theta_0)) \overline{Q} (s) = \mu_b \cdot e^{-r_{\text{disc}}} \left [ p - \frac{e^{c \iota}}{r_{\text{disc}}} \right ] \Gamma_1 (s) - \mu_b \left[ \frac{\alpha V B}{\mu_b \cdot m} - \frac{c}{r_{\text{disc}}} \right ] \Gamma_2 (s)
\]
(F23)
I show that
\[
\Gamma_1 (s) = \left[ N \left( -\ddot{a} (\theta_0) \sigma \sqrt{m} \right) + e^{\frac{1}{2} [(s + \ddot{a} (\theta_0))^2 - \ddot{a} (\theta_0)^2] \sigma^2 m} N \left( - (s + \ddot{a} (\theta_0)) \sigma \sqrt{m} \right) \right] \frac{1}{\gamma (\theta_0)} \left( \frac{1}{s - \eta (\theta_0)} - \frac{1}{s} \right)
\]
\[
- \left[ N \left( -\ddot{a} (\theta_0) \sigma \sqrt{m} \right) + e^{\frac{1}{2} [(s + \ddot{a} (\theta_0))^2 - \ddot{a} (\theta_0)^2] \sigma^2 m} N \left( 0 \right) \sigma \sqrt{m} \right] \frac{1}{\gamma (\theta_0)} \left( \frac{1}{s} - \frac{1}{s + \gamma (\theta_0)} \right)
\]
\[
+ \left[ N \left( \ddot{a} (\theta_0) \sigma \sqrt{m} \right) + e^{\frac{1}{2} [(s + \ddot{a} (\theta_0))^2 - \ddot{a} (\theta_0)^2] \sigma^2 m} N \left( 0 \right) \sigma \sqrt{m} \right] \times
\]
\[
\frac{1}{2} \ddot{a} (\theta_0) \eta (\theta_0) \left[ \frac{1}{s - \eta (\theta_0)} - \frac{1}{s + \gamma (\theta_0)} \right]
\]
\[
- \left[ N \left( \ddot{a} (\theta_0) \sigma \sqrt{m} \right) + e^{\frac{1}{2} [(s + \ddot{a} (\theta_0))^2 - \ddot{a} (\theta_0)^2] \sigma^2 m} N \left( 0 \right) \sigma \sqrt{m} \right] \times
\]
\[
\frac{1}{\gamma (\theta_0) - 2\ddot{a} (\theta_0)} \left[ \frac{1}{s + 2\ddot{a} (\theta_0)} - \frac{1}{s + \gamma (\theta_0)} \right]
\]
(F24)
and

\[\Gamma_2 (s) = \left[ N \left( \dot{z} (\theta) \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left( (s+\dot{\hat{\theta}}(\theta_0))^2 - \ddot{\dot{\theta}}(\theta)^2 \right) \sigma^2 m} N \left( (s + \dot{\hat{\theta}}(\theta_0)) \sigma \sqrt{m} \right) \right] \times \]

\[\times \frac{1}{\dot{a} (\theta_0) - \dot{\hat{\theta}} (\theta) + \eta (\theta_0)} \left( \frac{1}{s - \eta (\theta_0)} - \frac{1}{s + \dot{\hat{\theta}} (\theta) - \dot{\hat{\theta}} (\theta)} \right) \]

\[- \left[ N \left( \dot{z} (\theta) \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left( (s+\dot{\hat{\theta}}(\theta_0))^2 - \ddot{\dot{\theta}}(\theta)^2 \right) \sigma^2 m} N \left( (s + \dot{\hat{\theta}}(\theta_0)) \sigma \sqrt{m} \right) \right] \times \]

\[\times \frac{1}{\gamma (\theta_0) - \dot{a} (\theta_0) + \dot{\hat{\theta}}(\theta) + \eta (\theta_0)} \left( \frac{1}{s - \eta (\theta_0)} - \frac{1}{s + \dot{\hat{\theta}}(\theta) + \dot{\hat{\theta}}(\theta)} \right) \]

\[\left[ N \left( \dot{z} (\theta) \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left( (s+\dot{\hat{\theta}}(\theta_0))^2 - \ddot{\dot{\theta}}(\theta)^2 \right) \sigma^2 m} N \left( (s + \dot{\hat{\theta}}(\theta_0)) \sigma \sqrt{m} \right) \right] \times \]

\[\times \frac{1}{\gamma (\theta_0) - \dot{a} (\theta_0) - \ddot{z} (\theta)} \left( \frac{1}{s + \dot{\hat{\theta}}(\theta) + \dot{\hat{\theta}}(\theta)} - \frac{1}{s + \gamma (\theta_0)} \right) \]

\[\left( F 25 \right) \]

**Proof.** By the definition of \( F(\cdot , \cdot ; \theta_0) \) in equation \( C1 \),

\[ L \left[ F (m, v; \theta_0) \right] = L \left[ e^{-\theta v} N \left( h_1 (m, v; \theta_0) \right) \right] + L \left[ e^{-2\theta v} N \left( h_2 (m, v; \theta_0) \right) \right] \]

where

\[ h_1 (m, v; \theta_0) = \frac{(-v - \dot{a} (\theta_0) \sigma^2 m)}{\sigma \sqrt{m}} \]

\[ h_2 (m, v; \theta_0) = \frac{(-v - (-\dot{a} (\theta_0) \sigma^2 m))}{\sigma \sqrt{m}} \]

Applying the result in claim \( 3 \) in Appendix D.2,

\[ L \left[ F (m, v; \theta_0) \right] = \frac{1}{s} \left[ N \left( -\dot{\hat{\theta}}(\theta_0) \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left( (s+\dot{\hat{\theta}}(\theta_0))^2 - \ddot{\dot{\theta}}(\theta_0)^2 \right) \sigma^2 m} N \left( (s + \dot{\hat{\theta}}(\theta_0)) \sigma \sqrt{m} \right) \right] \]

\[+ \frac{1}{s + 2\dot{\hat{\theta}}(\theta_0)} \left[ N \left( \dot{\hat{\theta}}(\theta_0) \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left( (s+\dot{\hat{\theta}}(\theta_0))^2 - \ddot{\dot{\theta}}(\theta_0)^2 \right) \sigma^2 m} N \left( (s + \dot{\hat{\theta}}(\theta_0)) \sigma \sqrt{m} \right) \right] \]

Making use of the partial fraction technique in equation \( F16 \) (omitting \( \theta_0 \)),

\[ \Gamma_1 (s) = \left( \frac{1}{s - \eta} - \frac{1}{s + \gamma} \right) \frac{1}{s} \left[ N \left( -\dot{\theta} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left( (s+\dot{\hat{\theta}})^2 - \ddot{\theta}^2 \right) \sigma^2 m} N \left( (s + \dot{\hat{\theta}}) \sigma \sqrt{m} \right) \right] \]

\[\left( \frac{1}{s - \eta} - \frac{1}{s + \gamma} \right) \frac{1}{s + 2\dot{\hat{\theta}}(\theta_0)} \left[ N \left( \dot{\hat{\theta}}(\theta_0) \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left( (s+\dot{\hat{\theta}})^2 - \ddot{\theta}^2 \right) \sigma^2 m} N \left( (s + \dot{\hat{\theta}}) \sigma \sqrt{m} \right) \right] \]

\[= \left\{ \frac{1}{s - \eta} - \frac{1}{s + \gamma} \right\} \left\{ \frac{1}{\eta} - \frac{1}{\gamma - 2\dot{\hat{\theta}}} \right\} \left[ N \left( a \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left( (s+a)^2 - \dot{\theta}^2 \right) \sigma^2 m} N \left( (s + a) \sigma \sqrt{m} \right) \right] \]
Rearranging and collecting terms,
\[
\Gamma_1 (s) = \left[ N \left( -\hat{a} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{a}^2 \right] \sigma^2 m N \left( - (s + \hat{a}) \sigma \sqrt{m} \right) \right] \frac{1}{\eta} \left( \frac{1}{s - \eta} - \frac{1}{s} \right) \\
- \left[ N \left( -\hat{a} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{a}^2 \right] \sigma^2 m N \left( - (s + \hat{a}) \sigma \sqrt{m} \right) \right] \frac{1}{\gamma} \left( \frac{1}{s - \gamma} - \frac{1}{s + \gamma} \right) \\
+ \left[ N \left( \hat{a} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{a}^2 \right] \sigma^2 m N \left( - (s + \hat{a}) \sigma \sqrt{m} \right) \right] \frac{1}{2\hat{a} + \eta} \left( \frac{1}{s - \eta} - \frac{1}{s + 2\hat{a}} \right) \\
- \left[ N \left( \hat{a} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{a}^2 \right] \sigma^2 m N \left( - (s + \hat{a}) \sigma \sqrt{m} \right) \right] \frac{1}{\gamma - 2\hat{a}} \left( \frac{1}{s + 2\hat{a}} - \frac{1}{s + \gamma} \right)
\]

I proceed in a similar fashion with the function \( G (\cdot, \cdot; \theta) \). Applying the Laplace Transform to equation C9, we obtain
\[
L \left[ G (m, v; \theta) \right] = L \left[ e^{-s\hat{z}m} N (q_1 (m, v; \theta)) \right] + L \left[ e^{-s\hat{z}m} N (q_2 (m, v; \theta)) \right]
\]
where
\[
q_1 (m, v; \theta_0) = \frac{(-v - \hat{z} (\theta_0) \sigma^2 m)}{\sigma \sqrt{m}} \\
q_2 (m, v; \theta_0) = \frac{(-v - (-\hat{z} (\theta_0)) \sigma^2 m)}{\sigma \sqrt{m}}
\]
Another application of claim 3 in Appendix D.2 yields (omitting \( \theta \))
\[
L \left[ G (m, v; \theta) \right] = \frac{1}{s + (\hat{a} - \hat{z})} \left[ N \left( -\hat{z} \sigma \sqrt{m} \right) + \exp \left( \frac{1}{2} \left[ (s + (\hat{z} + (\hat{a} - \hat{z})))^2 - \hat{z}^2 \right] \sigma^2 m \right) \times \right] \\
\times N \left( - (s + \hat{z} + (\hat{a} - \hat{z})) \sigma \sqrt{m} \right) \\
+ \frac{1}{s + (\hat{a} + \hat{z})} \left[ N \left( -(-\hat{z}) \sigma \sqrt{m} \right) + \exp \left( \frac{1}{2} \left[ (s + (-\hat{z} + (\hat{a} + \hat{z})))^2 - (-\hat{z})^2 \right] \sigma^2 m \right) \times \right] \\
\times N \left( -(s + (-\hat{z} + (\hat{a} + \hat{z})) \sigma \sqrt{m} \right)
\]
Rearranging terms gives
\[
L \left[ G (m, v; \theta) \right] = \frac{1}{s + (\hat{a} (\theta_0) - \hat{z} (\theta))} \left[ N \left( -\hat{z} (\theta) \sigma \sqrt{m} \right) + \exp \left( \frac{1}{2} \left[ (s + \hat{z} (\theta_0))^2 - \hat{z} (\theta)^2 \right] \sigma^2 m \right) N \left( - (s + \hat{a} (\theta_0)) \sigma \sqrt{m} \right) \right] \\
+ \frac{1}{s + (\hat{a} (\theta_0) + \hat{z} (\theta))} \left[ N \left( \hat{z} (\theta) \sigma \sqrt{m} \right) + \exp \left( \frac{1}{2} \left[ (s + \hat{z} (\theta_0))^2 - \hat{z} (\theta)^2 \right] \sigma^2 m \right) N \left( - (s + \hat{a} (\theta_0)) \sigma \sqrt{m} \right) \right]
\]
By the partial fraction technique in equation F16
\[
\left( \frac{1}{s - \eta} - \frac{1}{s + \gamma} \right) \frac{1}{s + (\hat{a} - \hat{z})} = \frac{1}{s - \eta} - \frac{1}{s + \hat{a} - \hat{z}} - \frac{1}{\gamma - \hat{a} + \hat{z}} - \frac{1}{\gamma - \hat{a} - \hat{z}}
\]
and
\[
\left( \frac{1}{s - \eta} - \frac{1}{s + \gamma} \right) \frac{1}{s + (\hat{a} + \hat{z})} = \frac{1}{s - \eta} - \frac{1}{s + \hat{a} + \hat{z}} - \frac{1}{\gamma - \hat{a} + \hat{z}} - \frac{1}{\gamma - \hat{a} - \hat{z}}
\]
Therefore,

\[
\Gamma_2(s) = \left(\frac{1}{(s - \eta) - (s + \gamma)} - \frac{1}{s + (\hat{a} - \hat{z})}\right) \frac{1}{s + (\hat{a} - \hat{z})} \left[ N\left(\hat{\zeta}\sqrt{m}\right) + e^{\frac{1}{2}\left[(s + \hat{a})^2 - \hat{z}^2\right]}\sigma^2 N\left(- (s + \hat{a}) \sigma \sqrt{m}\right) \right] \\
+ \left(\frac{1}{(s - \eta) - (s + \gamma)} - \frac{1}{s + (\hat{a} - \hat{z})}\right) \frac{1}{s + (\hat{a} + \hat{z})} \left[ N\left(\hat{\zeta}\sqrt{m}\right) + e^{\frac{1}{2}\left[(s + \hat{a})^2 - \hat{z}^2\right]}\sigma^2 N\left(- (s + \hat{a}) \sigma \sqrt{m}\right) \right]
\]

\[
= \left\{ \frac{1}{s - \eta} - \frac{1}{s + \hat{a} - \hat{z}} \right\} \left(\frac{1}{\alpha - \hat{z} + \eta} - \frac{1}{\gamma - \hat{a} + \hat{z}}\right) \left[ N\left(\hat{\zeta}\sqrt{m}\right) + e^{\frac{1}{2}\left[(s + \hat{a})^2 - \hat{z}^2\right]}\sigma^2 N\left(- (s + \hat{a}) \sigma \sqrt{m}\right) \right]
\]

The result follows from rearranging terms in the expression above

\[
\Gamma_2(s) = \left[ N\left(\hat{\zeta}\sqrt{m}\right) + e^{\frac{1}{2}\left[(s + \hat{a})^2 - \hat{z}^2\right]}\sigma^2 N\left(- (s + \hat{a}) \sigma \sqrt{m}\right) \right] \frac{1}{\alpha - \hat{z} + \eta} \left(\frac{1}{s - \eta} - \frac{1}{s + \hat{a} - \hat{z}}\right)
\]

- \left[ N\left(\hat{\zeta}\sqrt{m}\right) + e^{\frac{1}{2}\left[(s + \hat{a})^2 - \hat{z}^2\right]}\sigma^2 N\left(- (s + \hat{a}) \sigma \sqrt{m}\right) \right] \frac{1}{\gamma - \hat{a} + \hat{z}} \left(\frac{1}{s + \hat{a} - \hat{z} - \frac{1}{s + \gamma}}\right)
\]

+ \left[ N\left(\hat{\zeta}\sqrt{m}\right) + e^{\frac{1}{2}\left[(s + \hat{a})^2 - \hat{z}^2\right]}\sigma^2 N\left(- (s + \hat{a}) \sigma \sqrt{m}\right) \right] \frac{1}{\hat{a} + \hat{z} + \eta} \left(\frac{1}{s - \eta} - \frac{1}{s + \hat{a} + \hat{z}}\right)
\]

- \left[ N\left(\hat{\zeta}\sqrt{m}\right) + e^{\frac{1}{2}\left[(s + \hat{a})^2 - \hat{z}^2\right]}\sigma^2 N\left(- (s + \hat{a}) \sigma \sqrt{m}\right) \right] \frac{1}{\gamma - \hat{a} - \hat{z}} \left(\frac{1}{s + \hat{a} + \hat{z} - \frac{1}{s + \gamma}}\right)
\]

□
Plugging the formulas in $F24$ and $F25$ into equation $F23$ and rearranging terms yield

\[
(\eta (\theta_0) + \gamma (\theta_0)) \bar{Q} (s) = \mu_b \cdot e^{-r_{\text{disc}} m} \left[ p - \frac{c}{r_{\text{disc}}} \right] \frac{1}{\eta (\theta_0)} \left( \frac{1}{s - \eta (\theta_0)} - \frac{1}{s} \right) \times \left[ N (-\hat{a} (\theta_0) \sigma \sqrt{m}) + e^{\frac{1}{2}[(s+\hat{a}(\theta_0))^2-\hat{a}(\theta_0)^2]\sigma^2 m N (- (s + \hat{a} (\theta_0)) \sigma \sqrt{m}) \right] 
- \mu_b \cdot e^{-r_{\text{disc}} m} \left[ p - \frac{c}{r_{\text{disc}}} \right] \frac{1}{\gamma (\theta_0)} \left( \frac{1}{s - \gamma (\theta_0)} - \frac{1}{s + \gamma (\theta_0)} \right) \times \left[ N (-\hat{a} (\theta_0) \sigma \sqrt{m}) + e^{\frac{1}{2}[(s+\hat{a}(\theta_0))^2-\hat{a}(\theta_0)^2]\sigma^2 m N (- (s + \hat{a} (\theta_0)) \sigma \sqrt{m}) \right] 
+ \mu_b \cdot e^{-r_{\text{disc}} m} \left[ p - \frac{c}{r_{\text{disc}}} \right] \frac{1}{2\hat{a} (\theta_0) + \eta (\theta_0)} \left( \frac{1}{s - \eta (\theta_0)} - \frac{1}{s + 2\hat{a} (\theta_0)} \right) \times \left[ N (\hat{a} (\theta_0) \sigma \sqrt{m}) + e^{\frac{1}{2}[(s+\hat{a}(\theta_0))^2-\hat{a}(\theta_0)^2]\sigma^2 m N (- (s + \hat{a} (\theta_0)) \sigma \sqrt{m}) \right] 
- \mu_b \cdot e^{-r_{\text{disc}} m} \left[ p - \frac{c}{r_{\text{disc}}} \right] \frac{1}{\gamma (\theta_0) - 2\hat{a} (\theta_0)} \left( \frac{1}{s + 2\hat{a} (\theta_0)} - \frac{1}{s + \gamma (\theta_0)} \right) \times \left[ N (\hat{a} (\theta_0) \sigma \sqrt{m}) + e^{\frac{1}{2}[(s+\hat{a}(\theta_0))^2-\hat{a}(\theta_0)^2]\sigma^2 m N (- (s + \hat{a} (\theta_0)) \sigma \sqrt{m}) \right] 
+ \mu_b \left[ \frac{\alpha V^B}{\mu_b \cdot m} - \frac{c}{r_{\text{disc}}} \right] \frac{1}{\hat{a} (\theta_0) - \hat{z} (\theta) + \eta (\theta)} \left( \frac{1}{s - \eta (\theta)} - \frac{1}{s + \hat{a} (\theta_0) - \hat{z} (\theta)} \right) \times \left[ N (-\hat{z} (\theta) \sigma \sqrt{m}) + e^{\frac{1}{2}[(s+\hat{a}(\theta_0))^2-\hat{z}(\theta)^2]\sigma^2 m N (- (s + \hat{a} (\theta_0)) \sigma \sqrt{m}) \right] 
+ \mu_b \left[ \frac{\alpha V^B}{\mu_b \cdot m} - \frac{c}{r_{\text{disc}}} \right] \frac{1}{\gamma (\theta_0) - \hat{a} (\theta_0) + \hat{z} (\theta)} \left( \frac{1}{s + \hat{a} (\theta_0) - \hat{z} (\theta)} - \frac{1}{s + \gamma (\theta_0)} \right) \times \left[ N (\hat{z} (\theta) \sigma \sqrt{m}) + e^{\frac{1}{2}[(s+\hat{a}(\theta_0))^2-\hat{z}(\theta)^2]\sigma^2 m N (- (s + \hat{a} (\theta_0)) \sigma \sqrt{m}) \right] 
- \mu_b \left[ \frac{\alpha V^B}{\mu_b \cdot m} - \frac{c}{r_{\text{disc}}} \right] \frac{1}{\hat{a} (\theta_0) + \hat{z} (\theta) + \eta (\theta)} \left( \frac{1}{s - \eta (\theta)} - \frac{1}{s + \hat{a} (\theta_0) + \hat{z} (\theta)} \right) \times \left[ N (\hat{z} (\theta) \sigma \sqrt{m}) + e^{\frac{1}{2}[(s+\hat{a}(\theta_0))^2-\hat{z}(\theta)^2]\sigma^2 m N (- (s + \hat{a} (\theta_0)) \sigma \sqrt{m}) \right] 
+ \mu_b \left[ \frac{\alpha V^B}{\mu_b \cdot m} - \frac{c}{r_{\text{disc}}} \right] \frac{1}{\gamma (\theta_0) - \hat{a} (\theta_0) - \hat{z} (\theta)} \left( \frac{1}{s + \hat{a} (\theta_0) + \hat{z} (\theta)} - \frac{1}{s + \gamma (\theta_0)} \right) \times \left[ N (\hat{z} (\theta) \sigma \sqrt{m}) + e^{\frac{1}{2}[(s+\hat{a}(\theta_0))^2-\hat{z}(\theta)^2]\sigma^2 m N (- (s + \hat{a} (\theta_0)) \sigma \sqrt{m}) \right] \right] \tag{F26}
\]

Define now the auxiliary function $M$ as

\[
M (v; x, w, p, q) \equiv L^{-1} \left\{ \left( \frac{1}{s + p} - \frac{1}{s + q} \right) \left[ N (w \sigma \sqrt{m}) + e^{\frac{1}{2}[(s+x)^2-w^2]\sigma^2 m N (- (s + x) \sigma \sqrt{m}) \right] \right\} \tag{F26}
\]

then

\[
M (v; x, w, p, q) = \left\{ N (w \sigma \sqrt{m}) - e^{\frac{1}{2}[(p-x)^2-w^2]\sigma^2 m N ( (p - x) \sigma \sqrt{m}) \right\} e^{-pv} 
+ e^{\frac{1}{2}[(p-x)^2-w^2]\sigma^2 m} e^{-pv} N \left( \frac{-v + (p - x) \sigma^2 m}{\sigma \sqrt{m}} \right) 
- \left\{ N (w \sigma \sqrt{m}) - e^{\frac{1}{2}[(q-x)^2-w^2]\sigma^2 m N ( (q - x) \sigma \sqrt{m}) \right\} e^{-qv} 
- e^{\frac{1}{2}[(q-x)^2-w^2]\sigma^2 m} e^{-qv} N \left( \frac{-v + (q - x) \sigma^2 m}{\sigma \sqrt{m}} \right) \right\} \tag{F27}
\]
\textbf{Proof.} Recall the definition of $\Delta (v; x, w, p)$ in Appendix D.2

$$
\Delta (v; x, w, p) \equiv L^{-1} \left\{ \frac{1}{s + p} \left[ N (w \sigma \sqrt{m}) + e^{\frac{1}{2} [(s + x)^2 - w^2]} \sigma^2 m N \left( - (s + x) \sigma \sqrt{m} \right) \right] \right\}
$$

This gives

$$
M (v; x, w, p, q) = \Delta (v; x, w, p) - \Delta (v; x, w, q)
$$

The result follows from a direct application of Lemma 8.

We then have

$$
M (v; x, w, p, x + w) = - K (v; x, w, q) \quad (F28)
$$

$$
M (v; x, w, p, x + w) = K (v; x, w, p) \quad (F29)
$$

where

$$
K (v; x, w, p) \equiv \left[ N (w \sigma \sqrt{m}) - e^{\frac{1}{2} [(p - x)^2 - w^2]} \sigma^2 m N \left( (p - x) \sigma \sqrt{m} \right) \right] e^{-pv}
$$

$$
+ e^{\frac{1}{2} [(p - x)^2 - w^2]} \sigma^2 m e^{-pv} N \left( \frac{-v + (p - x) \sigma^2 m}{\sigma \sqrt{m}} \right) - e^{-(x+w)v} N \left( \frac{-v + w \sigma^2 m}{\sigma \sqrt{m}} \right) \quad (F30)
$$

We are now able to compute the Laplace inverse of $\overline{Q} (s)$, $\mathbb{E}^\pi (v; \omega, \theta)^{28}$

$$
\frac{2}{\sigma^2} \mathbb{E}^\pi (v; \omega, \theta) = \frac{\mu_b \cdot e^{-r_{disc} m}}{\sigma^2 z (\theta_0)} \left[ \frac{1}{\mu (\theta_0)} K (v; \hat{\theta} (\theta_0), - \hat{\theta} (\theta_0), - \eta (\theta_0)) + \frac{1}{\mu (\theta_0)} K (v; \hat{\theta} (\theta_0), \hat{\theta} (\theta_0), - \eta (\theta_0)) + \frac{1}{\mu (\theta_0)} K (v; \hat{\theta} (\theta_0), \hat{\theta} (\theta_0), \eta (\theta_0)) \right]
$$

$$
+ \frac{\mu_b \left( \frac{\alpha V B}{\mu_b - \mu} - \frac{c}{r_{disc}} \right)}{\sigma^2 z (\theta_0)} \left[ - \frac{1}{\mu (\theta_0) - z (\theta)} \left( \frac{1}{z (\theta_0) - z (\theta)} - \frac{1}{z (\theta_0) + z (\theta)} \right) K (v; \hat{\theta} (\theta_0), \hat{\theta} (\theta_0), \hat{\theta} (\theta_0), - \eta (\theta_0)) \right] \quad (F31)
$$

\textbf{Proof.} For ease of exposition, I omit $\theta_0$ and $\theta$ and let

$$
\psi_1 \equiv \mu_b \cdot e^{-r_{disc} m} \left[ p - \frac{c}{r_{disc}} \right] \quad \psi_2 \equiv \mu_b \left( \frac{\alpha V B}{\mu_b - \mu} - \frac{c}{r_{disc}} \right)
$$

\footnote{Notice $\mathbb{E}^\pi (\cdot)$ does not depend on the risk-management cost parameter, $\iota$}

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By taking the inverse Laplace transformation of $\bar{Q}(s)$ in equation F26, we obtain

\[
(\eta + \gamma) E^\eta (\nu; \omega, \theta) = \psi_1 \frac{1}{\eta} L^{-1} \left[ \left( \frac{1}{s - \eta} - \frac{1}{s} \right) \left\{ N \left( -\hat{a} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{a}^2 \right] \sigma^2 m N \left( (s + \hat{a}) \sigma \sqrt{m} \right) \right\} \right]
- \psi_1 \frac{1}{\gamma} L^{-1} \left[ \left( \frac{1}{s - \eta} - \frac{1}{s + \gamma} \right) \left\{ N \left( -\hat{a} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{a}^2 \right] \sigma^2 m N \left( (s + \hat{a}) \sigma \sqrt{m} \right) \right\} \right]
+ \psi_1 \frac{1}{2\hat{a} + \eta} L^{-1} \left[ \left( \frac{1}{s - \eta} - \frac{1}{s + 2\hat{a}} \right) \left\{ N \left( \hat{a} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{a}^2 \right] \sigma^2 m N \left( (s + \hat{a}) \sigma \sqrt{m} \right) \right\} \right]
- \psi_1 \frac{1}{\gamma - 2\hat{a}} L^{-1} \left[ \left( \frac{1}{s - \eta} - \frac{1}{s + \gamma} \right) \left\{ N \left( \hat{a} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{a}^2 \right] \sigma^2 m N \left( (s + \hat{a}) \sigma \sqrt{m} \right) \right\} \right]
- \left[ \left( 1 \right) \left\{ N \left( -\hat{z} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{z}^2 \right] \sigma^2 m N \left( (s + \hat{a}) \sigma \sqrt{m} \right) \right\} \right]
+ \psi_2 \frac{1}{\gamma - \hat{a} + \hat{z}} L^{-1} \left[ \left( \frac{1}{s - \eta} - \frac{1}{s + \hat{a} - \hat{z}} \right) \left\{ N \left( -\hat{z} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{z}^2 \right] \sigma^2 m N \left( (s + \hat{a}) \sigma \sqrt{m} \right) \right\} \right]
- \psi_2 \frac{1}{\hat{a} + \hat{z} + \eta} L^{-1} \left[ \left( 1 \right) \left\{ N \left( \hat{z} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{z}^2 \right] \sigma^2 m N \left( (s + \hat{a}) \sigma \sqrt{m} \right) \right\} \right]
+ \psi_2 \frac{1}{\gamma - \hat{a} - \hat{z}} L^{-1} \left[ \left( 1 \right) \left\{ N \left( \hat{z} \sigma \sqrt{m} \right) + e^{\frac{1}{2} \left[ (s + \hat{a})^2 - \hat{z}^2 \right] \sigma^2 m N \left( (s + \hat{a}) \sigma \sqrt{m} \right) \right\} \right]
\]

By the formula for $M$ in equation F27, the expression above is equivalent to

\[
(\eta + \gamma) E^\eta (\nu; \omega, \theta) = \psi_1 \frac{1}{\eta} M (\nu; \hat{a}, -\hat{a}, -\eta, 0)
- \psi_1 \frac{1}{\gamma} M (\nu; \hat{a}, -\hat{a}, 0, \gamma)
+ \psi_1 \frac{1}{2\hat{a} + \eta} M (\nu; \hat{a}, \hat{a}, -\eta, 2\hat{a})
- \psi_1 \frac{1}{\gamma - 2\hat{a}} M (\nu; \hat{a}, \hat{a}, 2\hat{a}, \gamma)
- \psi_2 \frac{1}{\hat{a} - \hat{z} + \eta} M (\nu; \hat{a}, -\hat{z}, -\eta, \hat{a} - \hat{z})
+ \psi_2 \frac{1}{\gamma - \hat{a} + \hat{z}} M (\nu; \hat{a}, -\hat{z}, \hat{a} - \hat{z}, \gamma)
- \psi_2 \frac{1}{\hat{a} + \hat{z} + \eta} M (\nu; \hat{a}, \hat{z}, -\eta, \hat{a} + \hat{z})
+ \psi_2 \frac{1}{\gamma - \hat{a} - \hat{z}} M (\nu; \hat{a}, \hat{z}, \hat{a} + \hat{z}, \gamma)
\]
Next, by equations F28 and F29,

\[
\mathbb{E}^\sigma(v;\omega,\theta) = \frac{\psi_1}{\eta + \gamma} \left[ \frac{1}{\eta} K(v;\hat{a},-\hat{a},-\eta) + \frac{1}{\gamma} K(v;\hat{a},-\hat{a},\gamma) \right] + \frac{\psi_2}{\eta + \gamma} \left[ -\frac{1}{\hat{a}-\hat{z}+\eta} K(v;\hat{a},-\hat{z},-\eta) - \frac{1}{\hat{a}+\hat{z}+\eta} K(v;\hat{a},\hat{z},-\eta) \right]
\]

Finally, I invoke the results below, which follow from the definitions of \( \hat{a}, z, \hat{z}, \eta \) and \( \gamma \),

\[
\begin{align*}
\eta + \gamma &= 2z \\
2\hat{a} + \eta &= \hat{a} + z = \gamma \\
\gamma - 2\hat{a} &= z - \hat{a} = \eta \\
\hat{a} - \hat{z} + \eta &= z - \hat{z} \\
\gamma - \hat{a} + \hat{z} &= z + \hat{z} \\
\hat{a} + \hat{z} + \eta &= z + \hat{z} \\
\gamma - \hat{a} - \hat{z} &= z - \hat{z}
\end{align*}
\]

to get

\[
\mathbb{E}^\sigma(v;\omega,\theta) = \frac{1}{2} \mu_b \cdot e^{-r_{disc}m} \left[ p - \frac{c}{r_{disc}} \right] z \left[ \frac{1}{\eta} K(v;\hat{a},-\hat{a},-\eta) + \frac{1}{\gamma} K(v;\hat{a},-\hat{a},\gamma) \right] + \frac{1}{2} \left[ 1 - \frac{\alpha V_B}{\mu_b m} - \frac{c}{r_{disc}} \right] z \left[ -\frac{1}{\hat{a}-\hat{z}+\eta} K(v;\hat{a},-\hat{z},-\eta) - \frac{1}{\hat{a}+\hat{z}+\eta} K(v;\hat{a},\hat{z},-\eta) \right]
\]

Multiplying \( \mathbb{E}^\sigma(v;\omega,\theta) \) by \( 2/\sigma^2 \) yields the result.

\[\square\]

### F.7 The Equity Function - Preliminary I

Recall that \( Q(s) \) is the laplace transformation of \( \mathbb{E}^\sigma(v;\iota,\omega,\theta) \), and that we defined \( \hat{Q}(s) \) and \( \bar{Q}(s) \) in equations F20 and F21 so that

\[
\frac{1}{2} \sigma^2 Q(s) = \hat{Q}(s) + \bar{Q}(s)
\]

Therefore

\[
\mathbb{E}^\sigma(v;\iota,\omega,\theta) = \frac{2}{\sigma^2} L^{-1} \left[ \hat{Q}(s) \right] + \frac{2}{\sigma^2} L^{-1} \left[ \bar{Q}(s) \right] = \frac{2}{\sigma^2} \left( \mathbb{E}^\sigma(v;\iota,\omega,\theta) + \mathbb{E}^\sigma(v;\omega,\theta) \right)
\]
Combining the results in equations F22 and F31, we get

\[
\mathbb{E}^\mathcal{F}(v; t, \omega, \theta) = \left( \frac{\delta - t}{\nu - r_{\text{group}}} \right) V B^e v - \frac{\delta - t}{\sigma^2 z(\theta)} V B \left[ \frac{e^{\eta(\theta_0)v}}{\eta(\theta_0) - 1} + \frac{e^{-\gamma(\theta_0)v}}{\gamma(\theta_0) + 1} \right] + \frac{l}{2z(\theta)} \left( e^{\eta(\theta_0)v} - e^{-\gamma(\theta_0)v} \right) \\
+ \mu_b \left( 1 - \pi (c \cdot m) + (1 - e^{-r_{\text{discr}} m}) \right) \left( \eta(\theta_0) - 1 \right) \left( 1 - \frac{1}{\eta(\theta_0)} \right) \left( 1 - e^{-\gamma(\theta_0)v} \right) \\
+ \mu_b \left( \frac{e^{-r_{\text{discr}} m} [p - e^{-r_{\text{discr}} m}]}{\sigma^2 z(\theta)} \right) \left[ \frac{1}{\pi(\theta_0)} K(v; \hat{a}(\theta_0), -\hat{a}(\theta_0), -\eta(\theta_0)) + \frac{1}{\pi(\theta_0)} K(v; \hat{a}(\theta_0), -\hat{a}(\theta_0), \gamma(\theta_0)) \right] \\
- \mu_b \left( \frac{\bar{\alpha} \nu - \beta}{\sigma^2 z(\theta)} \right) \left[ \frac{1}{\pi(\theta_0) + \pi(\theta)} K(v; \hat{a}(\theta_0), -\hat{a}(\theta_0), -\eta(\theta_0)) + \frac{1}{\pi(\theta_0) + \pi(\theta)} K(v; \hat{a}(\theta_0), \hat{a}(\theta_0), \gamma(\theta_0)) \right]
\] (F32)

\section*{F.8 Imposing the $V \to \infty$ boundary condition}

As proved in claim 5, the equity value has to grow linearly when $V \to \infty$. Since $e^{\eta(\theta_0)v} = \left( \frac{V}{\theta_0} \right)^{\eta(\theta_0)}$ and $\eta(\theta_0) > 1$, to avoid explosion we require the coefficient on $e^{\eta(\theta_0)v}$ in $\mathbb{E}^\mathcal{F}(v; t, \omega, \theta)$ to collapse to zero.

Let $K^{[\eta]}$ denote the sum of the coefficients of $e^{\eta v}$ (alone) in $K$, and define the auxiliary function $k(v; x, w, p)$ as

\[
k(v; x, w, p) \equiv K(v; x, w, p) - e^{\eta v} K^{[\eta]}(v; x, w, p)
\] (F33)

We have

\[
K^{[\eta]}(v; \hat{a}(\theta_0), -\hat{a}(\theta_0), -\eta(\theta_0)) = N \left( -\hat{a}(\theta_0) \sigma \sqrt{m} \right) - e^{\eta m} N \left( -z(\theta_0) \sigma \sqrt{m} \right) 
\] (F34)
\[
K^{[\eta]}(v; \hat{a}(\theta_0), -\hat{a}(\theta_0), \gamma(\theta_0)) = 0
\] (F35)
\[
K^{[\eta]}(v; \hat{a}(\theta_0), \hat{a}(\theta_0), -\eta(\theta_0)) = N \left( -\hat{a}(\theta_0) \sigma \sqrt{m} \right) - e^{\eta m} N \left( -z(\theta_0) \sigma \sqrt{m} \right)
\] (F36)
\[
K^{[\eta]}(v; \hat{a}(\theta_0), \hat{a}(\theta_0), \gamma(\theta_0)) = 0
\] (F37)
\[
K^{[\eta]}(v; \hat{a}(\theta_0), -\hat{z}(\theta), -\eta(\theta_0)) = N \left( -\hat{z}(\theta) \sigma \sqrt{m} \right) - e^{\frac{1}{2}[z(\theta_0)^2 - \hat{z}(\theta)^2]2\sigma^2 m} N \left( -z(\theta_0) \sigma \sqrt{m} \right)
\] (F38)
\[
K^{[\eta]}(v; \hat{a}(\theta_0), -\hat{z}(\theta), \gamma(\theta_0)) = 0
\] (F39)
\[
K^{[\eta]}(v; \hat{a}(\theta_0), \hat{z}(\theta), -\eta(\theta_0)) = N \left( \hat{z}(\theta) \sigma \sqrt{m} \right) - e^{\frac{1}{2}[z(\theta_0)^2 - \hat{z}(\theta)^2]2\sigma^2 m} N \left( -z(\theta_0) \sigma \sqrt{m} \right)
\] (F40)
\[
K^{[\eta]}(v; \hat{a}(\theta_0), \hat{z}(\theta), \gamma(\theta_0)) = 0
\] (F41)

From the equations above, it follows that

\[
k(v; \hat{a}(\theta_0), w, -\eta(\theta_0)) = e^{\frac{1}{2}[z(\theta_0)^2 - w^2]2\sigma^2 m} e^{\eta v} N \left( -\frac{v - z(\theta_0) \sigma \sqrt{m}}{\sigma \sqrt{m}} \right) - e^{-(\hat{a}(\theta_0) + w) v} N \left( -\frac{v + w \sigma^2 m}{\sigma \sqrt{m}} \right)
\] (F42)
\[
k(v; \hat{a}(\theta_0), w, \gamma(\theta_0)) = K(v; \hat{a}(\theta_0), w, \gamma(\theta_0))
\] (F43)

for $w \in \left\{ -\hat{a}(\theta_0), \hat{a}(\theta_0), -\hat{z}(\theta), \hat{z}(\theta) \right\}$.

Proof. I start by recalling that $\gamma(\theta_0) = 2\hat{a}(\theta_0) + \eta(\theta_0)$ and

\[
\eta(\theta_0) + \hat{a}(\theta_0) = \gamma(\theta_0) - \hat{a}(\theta_0) = z(\theta_0)
\]
Moreover,

\[
\frac{1}{2} \left[ z (\theta_0)^2 - \hat{a} (\theta_0)^2 \right] \sigma^2 m = rm
\]

By the definition of \( K \) in equation F.30, we have (omitting \( \theta_0 \) and \( \theta \)):

\[
K (v; \hat{a}, w, -\eta) = \left[ N \left( w \sigma \sqrt{m} \right) - e^{\frac{1}{2} \left[ (\eta - \hat{a})^2 - w^2 \right] \sigma^2 m} \left( -\eta - \hat{a} \sigma \sqrt{m} \right) \right] e^{\eta v} \\
+ e^{\frac{1}{2} \left[ (\eta - \hat{a})^2 - w^2 \right] \sigma^2 m} e^{\eta v} N \left( \frac{-v + (\eta - \hat{a}) \sigma^2 m}{\sigma \sqrt{m}} \right) - \\
- e^{-(\hat{a} + w)v} N \left( \frac{-v + w \sigma^2 m}{\sigma \sqrt{m}} \right)
\]

Replacing \( -\eta - \hat{a} \) by \( -z \), we get

\[
K (v; \hat{a}, w, -\eta) = \left[ N \left( w \sigma \sqrt{m} \right) - e^{\frac{1}{2} \left[ z^2 - w^2 \right] \sigma^2 m} \left( -z \sigma \sqrt{m} \right) \right] e^{\eta v} \\
+ e^{\frac{1}{2} \left[ z^2 - w^2 \right] \sigma^2 m} e^{\eta v} N \left( \frac{-v - z \sigma^2 m}{\sigma \sqrt{m}} \right) - \\
- e^{-(\hat{a} + w)v} N \left( \frac{-v + w \sigma^2 m}{\sigma \sqrt{m}} \right)
\]

Therefore,

\[
K^{[\eta]} (v; \hat{a}, w, -\eta) = N \left( w \sigma \sqrt{m} \right) - e^{\frac{1}{2} \left[ z^2 - w^2 \right] \sigma^2 m} \left( -z \sigma \sqrt{m} \right) \\
k (v; \hat{a}, w, -\eta) = e^{\frac{1}{2} \left[ z^2 - w^2 \right] \sigma^2 m} e^{\eta v} N \left( \frac{-v - z \sigma^2 m}{\sigma \sqrt{m}} \right) - e^{-(\hat{a} + w)v} N \left( \frac{-v + w \sigma^2 m}{\sigma \sqrt{m}} \right)
\]

Specializing to the case where \( w = \pm \hat{a} \) yields

\[
K^{[\eta]} (v; \hat{a}, \pm \hat{a}, -\eta) = N \left( \pm \hat{a} \sigma \sqrt{m} \right) - e^{\frac{1}{2} \left[ (\pm \hat{a})^2 - (\pm \hat{a})^2 \right] \sigma^2 m} \left( -z \sigma \sqrt{m} \right) \\
= \left[ N \left( \pm \hat{a} \sigma \sqrt{m} \right) - e^{rm} N \left( -z \sigma \sqrt{m} \right) \right]
\]

and

\[
k (v; \hat{a}, \pm \hat{a}, -\eta) = e^{rm} e^{\eta v} N \left( \frac{-v - z \sigma^2 m}{\sigma \sqrt{m}} \right) - e^{-(\hat{a} \pm \hat{a})v} N \left( \frac{-v \pm \hat{a} \sigma^2 m}{\sigma \sqrt{m}} \right)
\]

For \( w = \pm \hat{z} \), \( K^{[\eta]} (v; \hat{a}, \pm \hat{z}, -\eta) = N \left( \pm \hat{z} \sigma \sqrt{m} \right) - e^{\frac{1}{2} \left[ \hat{z}^2 - \hat{z}^2 \right] \sigma^2 m} \left( -z \sigma \sqrt{m} \right) \)

and

\[
k (v; \hat{a}, \pm \hat{z}, -\eta) = e^{\frac{1}{2} \left[ \hat{z}^2 - \hat{z}^2 \right] \sigma^2 m} e^{\eta v} N \left( \frac{-v - z \sigma^2 m}{\sigma \sqrt{m}} \right) - e^{-(\hat{a} \pm \hat{z})v} N \left( \frac{-v \pm \hat{z} \sigma^2 m}{\sigma \sqrt{m}} \right)
\]

Consider now the case of \( K (v; \hat{a}, w, \gamma) \), for \( w \in \{ -\hat{a}, \hat{a}, \hat{z}, \hat{z} \} \):

\[
K (v; \hat{a}, w, \gamma) = \left[ N \left( w \sigma \sqrt{m} \right) - e^{\frac{1}{2} \left[ (\gamma - \hat{a})^2 - w^2 \right] \sigma^2 m} \left( (\gamma - \hat{a}) \sigma \sqrt{m} \right) \right] e^{-\gamma v} \\
+ e^{\frac{1}{2} \left[ (\gamma - \hat{a})^2 - w^2 \right] \sigma^2 m} e^{-\gamma v} N \left( \frac{-v + (\gamma - \hat{a}) \sigma^2 m}{\sigma \sqrt{m}} \right) - \\
- e^{-(\hat{a} + w)v} N \left( \frac{-v + w \sigma^2 m}{\sigma \sqrt{m}} \right)
\]

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Replacing \( \gamma - \hat{\alpha} \) by \( z \), we get

\[
K(v; \hat{\alpha}, w, \gamma) \equiv \left[ N \left( w\sigma\sqrt{m} \right) - e^{\frac{1}{2}[z^2-w^2]\sigma^2m} N \left( z\sigma\sqrt{m} \right) \right] e^{-\gamma v} + e^{\frac{1}{2}[z^2-w^2]\sigma^2m} e^{-\gamma v} N \left( \frac{-v + z\sigma^2m}{\sigma\sqrt{m}} \right) - e^{- (\hat{\alpha} + w) v} N \left( \frac{-v + w\sigma^2m}{\sigma\sqrt{m}} \right)
\]

Therefore,

\[
K^{[\eta]}(v; \hat{\alpha}, w, \gamma) = 0, \quad \text{for } w = -\hat{\alpha}, \hat{\alpha}, -\hat{z}, \hat{z}
\]

Collecting the coefficients of \( e^{\eta(\theta_0)v} \) in \( L^\eta(v; \theta, \theta) \) and setting the sum to zero, we have

\[
0 = -\frac{V^B}{\sigma^2 (z_0)} \frac{1}{\eta(\theta_0)} \left[ \frac{1}{\eta(\theta_0)} - 1 + \frac{l}{2z(\theta_0)} + \mu_b \left( 1 - \pi \right) (c \cdot m) + (1 - e^{-r_{\text{disc}} m}) \left( p - \frac{c}{r_{\text{disc}}} \right) \right] \frac{1}{\eta(\theta_0)} + \mu_b \cdot e^{-r_{\text{disc}} m} \left[ \frac{1}{\eta(\theta_0)} K^{[\eta]}(v; \hat{\alpha}(\theta_0), -\hat{\alpha}(\theta_0), -\eta(\theta_0)) + \frac{1}{\eta(\theta_0)} K^{[\eta]}(v; \hat{\alpha}(\theta_0), \hat{\alpha}(\theta_0), \gamma(\theta_0)) \right]
\]

Multiplying both sides by \( \sigma^2 z(\theta_0) \) and using the results in equations F34 to F41 yield

\[
0 = -\frac{V^B}{\eta(\theta_0)} + \mu_b \left( 1 - \pi \right) c \cdot m + (1 - e^{-r_{\text{disc}} m}) \left( p - \frac{c}{r_{\text{disc}}} \right) \frac{1}{\eta(\theta_0)} + \sigma^2 l
\]

Define the following auxiliary functions

\[
b(x; \theta_0) \equiv \frac{1}{z(\theta_0) + x} e^{-r_{\text{disc}} m} \left[ N \left( x\sigma\sqrt{m} \right) - e^{x m} N \left( -z(\theta_0) \sigma\sqrt{m} \right) \right]
\]

\[
B(x; \theta_0) \equiv \frac{1}{z(\theta_0) + x} \left[ N \left( x\sigma\sqrt{m} \right) - e^{\frac{1}{2} [z(\theta_0)^2 - x^2] \sigma^2 m} N \left( -z(\theta_0) \sigma\sqrt{m} \right) \right]
\]
Rewriting equation F44 in terms of the $b(\cdotp; \theta_0)$ and $B(\cdotp; \theta_0)$ functions and collecting the coefficients on $V_B$, we obtain

$$\frac{-\sigma^2}{2} l = - \left( \frac{\bar{\delta} - \bar{\iota}}{\eta(\theta_0) - 1} + \frac{\alpha}{m} [B(-\hat{z}(\theta); \theta_0) + B(\hat{z}(\theta); \theta_0)] \right) V_B + \mu_b \cdot \frac{(1 - \pi) c \cdot m + (1 - e^{-r_{disc}m}) (p - \frac{c}{\sigma z(\theta_0)})}{\eta(\theta_0)}$$

$$+ \mu_b \left( p - \frac{c}{\sigma z(\theta_0)} \right) [b(-\hat{a}(\theta_0); \theta_0) + b(\hat{a}(\theta_0); \theta_0)] + \mu_b \frac{c}{\sigma z(\theta_0)} [B(-\hat{z}(\theta); \theta_0) + B(\hat{z}(\theta); \theta_0)]$$

This equation can be used to solve for $V_B$ after we back out the value of $l$ by imposing the smooth-pasting condition.

**F.9 Imposing the Smooth-Pasting Condition**

Now that we have derived an expression for the equity value, we can impose the smooth-pasting condition to back out the value of $l$. The smooth-pasting condition ensures continuity of the equity function by requiring that the first-order derivative also converges to zero at the default barrier, that is

$$\lim_{V \downarrow V_B} \frac{\partial}{\partial V} E(\cdotp; \sigma, \eta, \theta) = 0$$

Rewriting it in terms of $v$ gives

$$\lim_{v \downarrow 0} \frac{\partial}{\partial v} E(\sigma; \cdotp; v, \eta, \theta) = l = 0$$

**Proof.** Since $\frac{dv}{dV} = 1/V = (1/V_B) e^{-v}$,

$$\frac{\partial}{\partial V} E(\cdotp; \sigma, \eta, \theta) = \frac{1}{V_B e^v} \frac{\partial}{\partial v} E(\sigma; \cdotp; v, \eta, \theta)$$

It follows that

$$\lim_{V \downarrow V_B} \frac{\partial}{\partial V} E(\cdotp; \sigma, \eta, \theta) = 0 \Leftrightarrow \lim_{v \downarrow 0} \frac{\partial}{\partial v} E(\sigma; \cdotp; v, \eta, \theta) = l = 0$$

**F.10 The Equity Function and the Optimal Default Barrier**

Imposing F47 and F48 on the formula for the equity function in equation F32, we get the following closed-form expression:

$$E(\sigma; \cdotp; \eta, \theta, \tau) = \left( \frac{\bar{\delta} - \bar{\iota}}{r_{\text{grow}}} \right) V_B e^v - \frac{(\bar{\delta} - \bar{\iota}) V_B^2}{\sigma^2 z(\theta_0) \gamma(\theta_0) + 1} + \mu_b \cdot \frac{(1 - \pi) c \cdot m + (1 - e^{-r_{disc}m}) (p - \frac{c}{\sigma z(\theta_0)})}{\eta(\theta_0)}$$

$$\left\{ \frac{1}{\eta(\theta_0)} + \frac{1 - e^{-\gamma(\theta_0)v}}{\gamma(\theta_0)} \right\}$$

$$+ \frac{\mu_b}{\sigma^2 z(\theta_0)} \left\{ e^{-r_{disc}m} \left( p - \frac{c}{\sigma z(\theta_0)} \right) A(\cdotp; \hat{a}(\theta_0), \theta_0) - \left( \frac{\alpha V_B}{\mu_b \cdot m} - \frac{c}{\sigma z(\theta_0)} \right) A(\cdotp; -\hat{z}(\theta), \theta_0) \right\}$$

(F49)
or, equivalently,

\[
E^\pi (v; \iota, \omega, \theta) = \mu_b \cdot \left\{ (\bar{\delta} - \iota) V^B \left[ \frac{e^v}{r - r_{grow}} - \frac{1}{\sigma^2 \gamma (\theta_0) + 1} e^{-\gamma (\theta_0) v} \right] \\
- \frac{(1 - \pi) c \cdot m + (1 - e^{-r_{disc}m}) \left( p - \frac{c}{r_{disc}} \right)}{\sigma^2 \gamma (\theta_0)} \left[ \frac{1}{\gamma (\theta_0)} + \frac{1 - e^{-\gamma (\theta_0) v}}{\gamma (\theta_0)} \right] \\
+ \frac{1}{\sigma^2 \gamma (\theta_0)} e^{-r_{disc}m} \left( p - \frac{c}{r_{disc}} \right) A (v; \hat{\theta}_b, \theta_0) - \left( \frac{\alpha V^B}{\mu_b \cdot m} - \frac{c}{r_{disc}} \right) A (v; -\hat{\theta}_b, \theta_0) \right\}
\]

where

\[
A (v; y, \theta_0) = \frac{1}{z (\theta_0) - y} \left[ K (v; \hat{\theta}_b, y, \gamma (\theta_0)) + k (v; \hat{\theta}_b, -y, -\eta (\theta_0)) \right] \\
+ \frac{1}{z (\theta_0) + y} \left[ K (v; \hat{\theta}_b, -y, \gamma (\theta_0)) + k (v; \hat{\theta}_b, y, -\eta (\theta_0)) \right]
\]

with \( K (v; x, w, p) \) and \( k (v; x, w, p) \) as defined in equations \( \text{F30} \) and \( \text{F33} \).

**Proof.** Recalling that \( \eta (\theta_0) = z (\theta_0) - \hat{\theta}_b (\theta_0) \) and \( \gamma (\theta_0) = z (\theta_0) + \hat{\theta}_b (\theta_0) \), and imposing \( \text{F43} \) and \( \text{F44} \) on equation \( \text{F32} \), we obtain (omitting \( \theta_0 \) and \( \theta \))

\[
E^\pi (v; \iota, \omega, \theta) = \left( \frac{\bar{\delta} - \iota}{r - r_{grow}} \right) V - \left( \frac{\bar{\delta} - \iota}{r_{grow}} \right) V^B e^{-\gamma v} \left[ \frac{1}{\gamma} + \frac{1 - e^{-\gamma v}}{\gamma} \right] \\
- \mu_b \left[ (1 - \pi) c \cdot m + (1 - e^{-r_{disc}m}) \left( p - \frac{c}{r_{disc}} \right) \right] \left[ \frac{1}{\eta (\theta_0)} + \frac{1 - e^{-\gamma (\theta_0) v}}{\eta (\theta_0)} \right] \\
+ \mu_b e^{-r_{disc}m} \left[ p - \frac{c}{r_{disc}} \right] \left[ \frac{1}{z + \eta (\theta_0)} k (v; \hat{\theta}_b, -\eta, -\eta) + \frac{1 - e^{-\gamma (\theta_0) v}}{z + \eta (\theta_0)} \right] K (v; \hat{\theta}_b, \gamma) \\
- \mu_b \left( \frac{\alpha V^B}{\mu_b \cdot m} - \frac{c}{r_{disc}} \right) \left[ -\frac{1}{z + \eta (\theta_0)} k (v; \hat{\theta}_b, -\eta, -\eta) - \frac{1 - e^{-\gamma (\theta_0) v}}{z + \eta (\theta_0)} \right] K (v; \hat{\theta}_b, \gamma) \\
\]

The result follows from applying the definition of \( A (v; y, \theta_0) \) in equation \( \text{F51} \) to the last two terms on the RHS of the expression above.

Finally, the optimal default barrier is obtained by setting \( l \) to zero in equation \( \text{F47} \) per equation \( \text{F48} \).

\[\text{Notice } K (v; \hat{\theta}_b, y, -\eta (\theta_0)) \text{ is related to } k (v; \hat{\theta}_b, -y, -\eta (\theta_0)) \text{ according to} \]

\[
K (v; \hat{\theta}_b, y, -\eta (\theta_0)) = e^{\eta (\theta_0) v} \left[ N \left( y \sigma \sqrt{m} \right) - e^{\frac{1}{2} \left[ (\theta_0)^2 - y^2 \right]} \sigma^2 \cdot m N \left( -z (\theta_0) \sigma \sqrt{m} \right) \right] \\
+ k (v; \hat{\theta}_b, -y, -\eta (\theta_0))
\]
THEOREM 4: The endogenous bankruptcy boundary $V^B$ is given by

$$
= \mu_b \left\{ \left(1 - \pi \right) c \cdot m + (1 - e^{-r_{disc}m}) \left( p - \frac{c}{r_{disc}} \right) \right\} + \left( p - \frac{c}{r_{disc}} \right) \left\{ \left( \frac{g}{\eta} (\theta_0) - \delta - \lambda \right) + \alpha \left( B(-\lambda; \theta_0) + B(\lambda; \theta_0) \right) \right\}^{-1}
$$

where $r_{grow} \equiv r - \delta$, $r_{disc} \equiv r + \xi \kappa$, $\theta_0 \equiv (r_{grow}, \sigma)$, $\theta \equiv (r_{disc}, r_{grow}, \sigma)$, $\omega \equiv (\mu_b, m, c, p)$ and

$$
\hat{a}(\theta_0) = \frac{r_{grow} - \frac{1}{2} \sigma^2}{\sigma^2}, \quad z(\theta_0) = \frac{\hat{a}(\theta_0)^2 \sigma^4 + 2 \sigma^2 r}{\sigma^2}, \quad \hat{z}(\theta) = \frac{\hat{a}(\theta_0)^2 \sigma^4 + 2 \sigma^2 r_{disc}}{\sigma^2}
$$

$\eta(\theta_0) = z(\theta_0) - \hat{a}(\theta_0) > 1$

$$
b(x; \theta_0) = \frac{1}{z(\theta_0) + x} e^{-r_{disc}m} \left[ N(x \sigma \sqrt{m}) - e^{r_{disc}m} N(-z(\theta_0) \sigma \sqrt{m}) \right]
$$

$B(x; \theta_0) = \frac{1}{z(\theta_0) + x} \left[ N(x \sigma \sqrt{m}) - e^{x^2/2} z(\theta_0)^2 \sigma^2 N(-z(\theta_0) \sigma \sqrt{m}) \right]
$$

and $N(\cdot)$ is the cumulative normal distribution.

Appendix G. Pre-Volatility Shock Equity Value

In this section, I show how to solve for the equity function of pre-volatility-shock firms using the Finite Difference Method.

G.1 Equity Value Partial Differential Equation

Let $E^\lambda(V; \lambda, \omega, \theta)$ denote the equity value function prior to the volatility shock when the firm chooses not to manage risk, where $\omega \equiv (\mu_b, m, c, p)$ and $\theta \equiv (r_{disc}, r_{grow}, \sigma_l, \sigma_h)$. I will show that the firm’s equity value must satisfy the following PDE:

$$(r + \lambda)E^\lambda(V; \lambda, \omega, \theta) = \delta V_t - (1 - \pi)(\mu_b \cdot c \cdot m) + \mu_b \left[ d(V_t, m; V^B, \lambda, \omega, \theta) - p \right] + \lambda E^\sigma(V_t; \omega, \theta_t) + (r - \delta) V_t E^\lambda(V_t; \lambda, \omega, \theta_t) + \frac{1}{2} \sigma_l^2 V_t^2 E^\lambda(V_t; \lambda, \omega, \theta_t)
$$

where $d(V_t, m; V^B, \lambda, \omega, \theta)$ is the pre-volatility shock bond price defined in Theorem 2, $E^\sigma(V_t; \omega, \theta_t)$ is the constant volatility equity value derived in the previous section, $\theta_t \equiv (r_{disc}, r_{grow}, \sigma_h)$, with $\sigma_h > \sigma_l$, and $V^B_t$ is the pre-volatility shock default barrier.

Proof. Recall the proof of equation $F_2$. As before, at any time $t$, the firm’s instantaneous net cash-flow is the sum of its net dividends and debt rollover gains or losses, minus its after-tax coupon payments. Omitting $dt$, net cash-flows are as follows:

$$
\varphi(V_t; \lambda, \omega, \theta) \equiv \frac{\delta V_t}{\text{dividends}} - (1 - \pi)(\mu_b \cdot c \cdot m) + \mu_b \left[ d(V_t, m; V^B, \lambda, \omega, \theta) - p \right]
$$

rollover gain/loss
At the arrival of the first volatility shock, the firm’s volatility is immediately and permanently increased to $\sigma_h$. The value of equity is then given by $E^\sigma(V_t; \omega, \theta_h)$, defined in Theorem 3. By equation D1, the time-$t$ conditional probability of the first-volatility shock arriving in the next interval of time $dt$ is simply $1_{\{\sigma > t\}} (1 - e^{-\lambda dt})$. Therefore, the equity value Bellman equation is

$$E^\lambda(V_t; \lambda, \omega, \theta) = \lim_{dt \downarrow 0} E^Q \left[ \int_t^{t+dt} e^{-rs} \varphi(V_s; \lambda, \omega, \theta) ds + e^{-\lambda dt} E^\lambda(V_{t+dt}; \lambda, \omega, \theta) + \left(1 - e^{-\lambda dt}\right) E^\sigma(V_{t+dt}; \omega, \theta_h) \right] \left| \mathcal{F}_t \right]$$

Approximating the integral on the RHS by $e^{-rdt} \varphi(V_t) dt$, rearranging terms and dividing by $e^{-rdt}$, we get

$$0 = \lim_{dt \downarrow 0} E^Q \left[ \varphi(V_t; \lambda, \omega, \theta) dt + \lambda dt E^\sigma(V_{t+dt}, \omega, \theta_h) + e^{-\lambda dt} E^\lambda(V_{t+dt}; \lambda, \omega, \theta) - e^{rdt} E^\lambda(V_t; \lambda, \omega, \theta) \right] \left| \mathcal{F}_t \right]$$

Approximating $e^{-\lambda dt}$ and $e^{rdt}$ by $1 - \lambda dt$ and $1 + r dt$, respectively, gives

$$0 = \lim_{dt \downarrow 0} E^Q \left[ \varphi(V_t; \lambda, \omega, \theta) dt + \lambda dt E^\sigma(V_{t+dt}, \omega, \theta_h) + (1 - \lambda dt) E^\lambda(V_{t+dt}; \lambda, \omega, \theta) - (1 + r dt) E^\lambda(V_t; \lambda, \omega, \theta) \right] \left| \mathcal{F}_t \right]$$

Dividing by $dt$ and taking limits yields

$$0 = \varphi(V_t; \lambda, \omega, \theta) + \lambda \left[ -E^\lambda(V_{t+dt}; \lambda, \omega, \theta) + E^\sigma(V_{t+dt}, \omega, \theta_h) \right] - r E^\lambda(V_t; \lambda, \omega, \theta)$$

By Ito’s Lemma (omitting $\lambda$, $\omega$ and $\theta$):

$$\lim_{dt \downarrow 0} \frac{1}{dt} \cdot E^Q \left[ E^\lambda(V_{t+dt}) - E^\lambda(V_t) \right] \left| \mathcal{F}_t \right] = \lim_{dt \downarrow 0} \frac{1}{dt} \cdot E^Q \left[ E^\lambda(V_t) + \frac{\partial E^\lambda(V_t)}{\partial V} dV_t + \frac{1}{2} \frac{\partial^2 E^\lambda(V_t)}{\partial V^2} (dV_t)^2 - E^\lambda(V_t) \right] \left| \mathcal{F}_t \right]$$

$$= \lim_{dt \downarrow 0} \frac{1}{dt} \cdot E^Q \left[ \frac{\partial E^\lambda(V_t)}{\partial V} \left( r_{grow} V_t dt + \sigma_t V_t dZ_t \right) + \frac{1}{2} \frac{\partial^2 E^\lambda(V_t)}{\partial V^2} \sigma_t^2 V_t^2 dt \right] \left| \mathcal{F}_t \right]$$

$$= r_{grow} V_t E^\lambda(V_t) + \frac{1}{2} \sigma_t^2 V_t^2 E_{VV}^\lambda(V_t)$$

Finally, by combining the last two equations and setting $r_{grow}$ to $r - \delta$, we arrive at the solution.

As in the derivations of the equity function when volatility is constant, instead of working with the underlying value of assets $V$ directly, I resort to the transformed variable $v \equiv \ln(V/V_i^\mu)$ to solve a constant coefficient PDE. Let $\tilde{E}^\lambda(v)$ denote the pre-volatility shock equity value as a function of $v \equiv \ln(V/V_i^\mu)$, so that $E^\lambda(v) = E^\lambda(V)$. Rewriting the equity PDE in terms of $v$, we obtain

$$(r + \lambda) \tilde{E}^\lambda(v_t; \lambda, \omega, \theta) = \tilde{V}_i^B e^v_t + \mu_t \left[ - (1 - \pi) \circ m + d^\lambda \left( V_i^B e^{v_t}, m; V_i^B, \lambda, \omega, \theta \right) - p \right]$$

$$+ \lambda E^\sigma(V_i^B e^{v_t}; \omega, \theta_h) + \left( r - \delta - \frac{1}{2} \sigma_t^2 \right) E^\lambda(v_t; \lambda, \omega, \theta) + \frac{1}{2} \sigma_t^2 \tilde{E}_{vv}^\lambda(v_t; \lambda, \omega, \theta)$$

(G2)
Proof. Notice that \( V_t = V_t^B e^{\mu t} \) and \( \partial \psi / \partial V = 1/V \). Deriving \( E^\lambda \) with respect to \( v \) gives (omitting \( \lambda, \omega \) and \( \theta_l \)):

\[
E^\lambda_V(V) = \frac{1}{V} \hat{E}^\lambda_v(v)
\]

Differentiating again yields

\[
E^\lambda_{VV}(V) = \frac{1}{V^2} \left[ \hat{E}^\lambda_{vv}(v) - \frac{1}{2} \hat{E}^\lambda_v(v) \right]
\]

The result follows from plugging the expressions above into equation G1.

Limited liability implies that the value of equity is zero at the default boundary, that is \( E^\lambda(V_B) = 0 \), or equivalently \( \hat{E}^\lambda(0) = 0 \).

G.2 Finite-Differences Method for the Computation of Equity Value

The equity value PDE in equation G2 is of the form

\[
r^e_{\text{disc}} \hat{E}^\lambda(v; \lambda, \omega, \theta) = \psi(v; \lambda, \omega, \theta) + \nu \hat{E}^\lambda_v(v; \lambda, \omega, \theta) + \frac{1}{2} \sigma^2 \hat{E}^\lambda_{vv}(v; \lambda, \omega, \theta)
\]

where \( \nu \in \mathbb{R}^* \) and the superscript \( e \) in \( r^e_{\text{disc}} \) stands for equity and serves to distinguish the discount rate in the equity PDE from the bond investors’ discount rate, \( r^i_{\text{disc}} = r + \xi \kappa \).

This type of PDE can be solved via the Finite Differences Method. To do so, I discretize \( v \) by forming a grid of size \( N, v = \{ v_0, v_1, v_2, \ldots, v_N \} \), where \( v_0 = 0 \). First- and second-order derivatives are approximated by the following central-difference formulas:

\[
\frac{\partial \hat{E}^\lambda_j}{\partial v} = \frac{\hat{E}^\lambda_{j+1} - \hat{E}^\lambda_{j-1}}{2\Delta x}
\]

\[
\frac{\partial^2 \hat{E}^\lambda_j}{\partial v^2} = \frac{\hat{E}^\lambda_{j+1} - 2\hat{E}^\lambda_j + \hat{E}^\lambda_{j-1}}{(\Delta x)^2}
\]

where superscript \( j \) stands for gridpoint \( v_j \), for \( j = 1, \ldots, N - 1 \). Plugging these formulas into the PDE, we obtain

\[
r^e_{\text{disc}} \hat{E}^\lambda_j = \psi_j + \nu \left( \frac{\hat{E}^\lambda_{j+1} - \hat{E}^\lambda_{j-1}}{2\Delta x} \right) + \frac{1}{2} \sigma^2 \left( \frac{\hat{E}^\lambda_{j+1} - 2\hat{E}^\lambda_j + \hat{E}^\lambda_{j-1}}{(\Delta x)^2} \right)
\]

Rearranging,

\[
0 = \left( \frac{\nu}{2\Delta x} + \frac{1}{2} \frac{\sigma^2}{(\Delta x)^2} \right) \hat{E}^\lambda_{j+1} + \left( -r^e_{\text{disc}} - \frac{\sigma^2}{(\Delta x)^2} \right) \hat{E}^\lambda_j + \left( -\frac{\nu}{2\Delta x} + \frac{1}{2} \frac{\sigma^2}{(\Delta x)^2} \right) \hat{E}^\lambda_{j-1} + \psi_j \quad (G3)
\]

Define the constants

\[
qu \equiv \frac{1}{2} \left( \frac{\nu}{\Delta x} + \frac{\sigma^2}{(\Delta x)^2} \right)
\]

\[
qm \equiv - \left( r^e_{\text{disc}} + \frac{\sigma^2}{(\Delta x)^2} \right)
\]

\[
qd \equiv \frac{1}{2} \left( -\frac{\nu}{\Delta x} + \frac{\sigma^2}{(\Delta x)^2} \right)
\]
and the vector of coefficients

$$ \mathbf{q} \equiv (q_u, q_m, q_d) $$

The system of discretized PDEs given by equation (G3) can be written in matrix form as follows

$$ \begin{pmatrix} q_u & q_m & q_d \\ \equiv & \equiv & \equiv \mathbf{q} \end{pmatrix} \begin{bmatrix} \bar{E}_{j+1}^\lambda \\ \bar{E}_j^\lambda \\ \bar{E}_{j-1}^\lambda \end{bmatrix} + \psi_j = 0, \quad j = 1, \ldots, N - 1 $$

(G4)

The PDEs involving the end points of the v-grid, that is $j = 0$ and $j = N$, are added to the system above by invoking the boundary conditions.

G.2.1 The Upper Boundary Condition

As in the baseline model case, I first establish the limiting value of the risky bond as $V$ increases. Next, I replace the risky bond price for this value in the partial differential equation G1, and solve for the limiting equity price as a function of $V$.

CLAIM 8: The pre-volatility shock bond price $d^\lambda (V_t, \tau; V_t^B, \lambda, \omega, \theta)$ converges to that of a credit-risk-free bond, $b^{crf} (\tau; r_{disc}^i, \omega)$, as $V_t \to \infty.$

$$ \lim_{V_t \to \infty} d^\lambda (V_t, \tau; V_t^B, \lambda, \omega, \theta) = b^{crf} (\tau; r_{disc}^i, \omega), \quad \tau \in [0, m], \ V_t^B > 0 $$

where $b^{crf} (\tau; r_{disc}^i, \omega) = \frac{c}{r_{disc}^i} + e^{-r_{disc}^i \tau} \left[ p - \frac{c}{r_{disc}^i} \right]$, as defined in equation (F7).

Proof. Recall, from the proof of claim 4, that

$$ \lim_{V_t \to \infty} F \left( \tau, \ln \left( \frac{V_t}{V_B} \right); \theta_l \right) = 0 $$

and

$$ \lim_{V_t \to \infty} G \left( \tau, \ln \left( \frac{V_t}{V_B} \right); \theta_l \right) = 0 $$

Applying these limits to the pre-volatility-shock bond pricing formula in Theorem 2, we obtain

$$ \lim_{V_t \to \infty} d^\lambda (V_t, \tau; V_t^B, \lambda, \omega, \theta) = \frac{c}{r_{disc}^i + \lambda} + e^{-r_{disc}^i \lambda \tau} \left[ p - \frac{c}{r_{disc}^i + \lambda} \right] $$

$$ + \lim_{V_t \to \infty} \lambda \int_t^{t+\tau} e^{-r_{disc}^i \lambda \tau} \left\{ \int_{V_B}^{\infty} d^\tau (V_t - s; V_B^B (\omega, \theta_h), \omega, \theta_h) \times \right\} $$

$$ \times \left[ -\frac{\partial}{\partial V} \psi_{v,t} (s - t, \ln \left( \frac{V_t}{V_B} \right); \ln \left( \frac{V}{V_B (\omega, \theta_h)} \right), \theta_l) \right] dV \right\} ds $$

From the proof of Lemma 7, we know that the inner integral on the RHS equals the time-$t$ expectation of the bond price payoff when volatility shock happens at time $s < t$ before default.

$$ E^Q \left[ d^\tau (V_t, \tau; V_B^B (\omega, \sigma_h), \omega, \theta_h) 1_{\{d_{s,t}^i > s\}} \big| \mathcal{F}_t \right], \quad s \in [t, t + \tau] $$

Therefore, we can replace the second the double-integral term with

$$ \lim_{V_t \to \infty} \lambda \int_t^{t+\tau} e^{-r_{disc}^i \lambda \tau} E^Q \left[ d^\tau (V_t - (s - t); V_B^B (\sigma_h), \omega, \theta_h) 1_{\{d_{s,t}^i > s\}} \big| \mathcal{F}_t \right] ds $$
Furthermore, since \( d\sigma(\cdot, \tau; V^B, \omega, \theta_h) 1_{\{t^B > \tau\}} \) satisfies standard regularity conditions, we can interchange the limit with the integral and expectation to compute

\[
\lambda \int_t^{t+\tau} e^{-(r^i_{\text{disc}}+\lambda)(s-t)} E^Q \left[ \lim_{V_i \uparrow \infty} \left\{ d\sigma (V, \tau - (s-t); V^B, (\sigma_h), \omega, \theta_h) 1_{\{t^B > \tau\}} \right\} | \mathcal{F}_t \right] \, ds
\]

By claim 4, the bond price inside the expectation on the RHS converges to the value of the credit-risk-free bond. Moreover, \( 1_{\{t^B > \tau\}} \rightarrow 1 \) as the credit risk vanishes. Thus, the expectation term boils down to \( b^{\sigma f} (\tau - (s-t); r^i_{\text{disc}}, \omega) \) and we are left with

\[
\lambda \int_t^{t+\tau} e^{-(r^i_{\text{disc}}+\lambda)(s-t)} b^{\sigma f} (\tau - (s-t); r^i_{\text{disc}}, \omega) \, ds
\]

By the formula for \( b^{\sigma f} (\tau; r^i_{\text{disc}}, \omega) \) in equation F7,

\[
\lambda \int_t^{t+\tau} e^{-(r^i_{\text{disc}}+\lambda)(s-t)} b^{\sigma f} (\tau - (s-t); r^i_{\text{disc}}, \omega) \, ds = \frac{c}{r^i_{\text{disc}} + \lambda} \left\{ \left( e^{-(r^i_{\text{disc}}+\lambda)\tau} - e^{-r^i_{\text{disc}}\tau} \right) + \frac{\lambda}{r^i_{\text{disc}}} \left( 1 - e^{-r^i_{\text{disc}}\tau} \right) \right\}
\]

Hence, the pre-volatility bond price converges to

\[
\lim_{V_i \uparrow \infty} d\lambda (V_t, \tau; V^B_t, \lambda, \omega, \theta) = \frac{c}{r^i_{\text{disc}} + \lambda} + e^{-(r^i_{\text{disc}}+\lambda)\tau} \left[ p - \frac{c}{r^i_{\text{disc}} + \lambda} \right] + \frac{c}{r^i_{\text{disc}} + \lambda} \left\{ \left( e^{-(r^i_{\text{disc}}+\lambda)\tau} - e^{-r^i_{\text{disc}}\tau} \right) + \frac{\lambda}{r^i_{\text{disc}}} \left( 1 - e^{-r^i_{\text{disc}}\tau} \right) \right\}
\]

Q.E.D.

Replacing \( d\lambda (V_t, \tau; V^B_t, \lambda, \omega, \theta) \) with \( b^{\sigma f} (\tau; r^i_{\text{disc}}, \omega) \) in equation F7, we can now show that the pre-volatility-shock equity function converges to a linear function as in claim G1.

CLAIM 9: As \( V \) increases, the pre-volatility-shock equity function converges to the difference between the underlying value of assets and the present value of the coupon payments and rollover costs of a credit-risk-free debt\(^{30}\):

\[
E^\lambda (V_t; \lambda, \omega, \theta) \rightarrow V + \mu_b \frac{-(1-\pi)c \cdot m + b^{\sigma f}(m; r^i_{\text{disc}}, \omega) - p}{r}, \quad \text{as } V \rightarrow \infty
\]

Proof. By claim 5, as \( V \rightarrow \infty \), we can replace \( E^\sigma (V_t; 0, \omega, \theta_h) \) by its limiting function

\[
E^\sigma (V_t; 0, \omega, \theta_h) \rightarrow \phi_1 (0) V + \phi_0
\]

where

\[
\phi_0 = \frac{H}{r} = \frac{-(1-\pi)(\mu_b \cdot c \cdot m) + \mu_b \left[ b^{\sigma f}(m; r^i_{\text{disc}}, \omega) - p \right]}{r}
\]

\[
\phi_1 (t) = \frac{\overline{r}}{r - r_{\text{grow}}}
\]

\(^{30}\)This is the same limiting function obtained in the baseline case when risk-management costs are zero, as stated in corollary 3.
and \( r_{\text{grow}} = r - \bar{\delta} \) (which gives \( \phi_1(0) = 1 \)).

Additionally, claim 8 allows us to substitute \( b^{crf}(m; r_{\text{disc}}^i, \omega) \) for \( d^\lambda(V_t, m; V_t^B, \lambda, \omega, \theta) \). Therefore, as \( V \to \infty \), equation G1 becomes

\[
(r + \lambda) E^\lambda(V; \lambda, \omega, \theta) = \lambda [\phi_1(0) V + \phi_0] + (r - \bar{\delta}) V_t E^\lambda(V_t; \lambda, \omega, \theta) + \frac{1}{2} \sigma_t^2 V_t^2 E^\lambda_{VV}(V_t; \lambda, \omega, \theta)
\]

\[
+ 3 V_t + \left[-(1 - \pi) (\mu_b \cdot c \cdot m) + \mu_b \left[ b^{crf}(m; r_{\text{disc}}^i, \omega) - p \right] \right] \equiv H
\]

Let the limiting polynomial be

\[
\hat{p}(V) = \sum_{j=0}^{n} \hat{\phi}_j V^j, \quad n > 0
\]

Replacing \( E^\lambda \) with \( \hat{p}(\cdot) \) gives

\[
(r + \lambda) \sum_{j=0}^{n} \hat{\phi}_j V^j = \lambda [\phi_1(0) V + \phi_0] + \bar{\delta} V_t + H
\]

\[
+ (r - \bar{\delta}) V \left[ \sum_{j=1}^{n} \hat{\phi}_j V^{j-1} \right] + \frac{1}{2} \sigma_t^2 V^2 \left[ \sum_{j=2}^{n} \hat{\phi}_j V^{j-2} \right]
\]

Rearranging and collecting terms, we get

\[
(r + \lambda) \hat{\phi}_0 - \lambda \hat{\phi}_0 - H = - \left[ (\lambda + \bar{\delta}) \hat{\phi}_1 - \lambda \phi_1(0) - \bar{\delta} \right] V + \left( \lambda - \bar{\delta} + \frac{1}{2} \sigma_t^2 \right) \left[ \sum_{j=2}^{n} \hat{\phi}_j V^{j-2} \right]
\]

Since the equation above must be valid for all sufficiently large values of \( V \), the coefficients of \( p \) on all terms of degree higher than one must be zero

\[
\hat{\phi}_j = 0, \quad \forall j \geq 2
\]

Moreover,

\[
(r + \lambda) \hat{\phi}_0 + \lambda \hat{\phi}_0 - H = 0
\]

and

\[
(\lambda + \bar{\delta}) \hat{\phi}_1 - \lambda \phi_1(0) - \bar{\delta} = 0
\]

Solving for \( \hat{\phi}_0 \) and \( \hat{\phi}_1 \) gives

\[
\hat{\phi}_0 = \frac{H + \lambda \phi_0}{r + \lambda} = \frac{H}{r} = \phi_0
\]

\[
\hat{\phi}_1 = \frac{\lambda \phi_1(0) + \bar{\delta}}{\lambda + \bar{\delta}} = 1
\]

Therefore, as \( V \) increases, the pre-volatility-shock equity function converges to \( V + \phi_0 \). Q.E.D. \( \Box \)
By claim 9, we can approximate the value of $\hat{E}^\lambda(v)$ in the upper barrier by the value of the credit-risk-free equity:

$$E^{crf}(v; V^B_t, \omega) \equiv V^B_t e^v + \mu_b \frac{-(1-\pi)c \cdot m + b^{crf}(m; r^i_{disc}, \omega) - p}{r}$$

that is,

$$\hat{E}^\lambda_N = \hat{E}^\lambda(v_N) \approx E^{crf}(v_N; V^B_t, \omega)$$

Plugging the expression above into equation G4 evaluated at $j = N - 1$ yields

$$\begin{bmatrix} q_u & q_m & q_d \end{bmatrix} \begin{bmatrix} E^{crf}(v_N; V^B_t , \omega) \\ \hat{E}_N \\ \hat{E}_{N-1} \\ \hat{E}_{N-2} \end{bmatrix} + \psi_{N-1} = 0$$

(G5)

G.2.2 The Lower Boundary Condition

To establish the lower boundary condition, recall that all outstanding debt comes due at default and that the residual assets must be used first to cover debt repayment expenses (absolute priority rule.) The book value of debt is given by

$$D^{crf}(r^i_{disc}, \omega) = \int_0^m b^{crf}(\tau; r^i_{disc}, \omega) \cdot \mu_b \cdot d\tau$$

$$= \mu_b \left[ \frac{c \cdot m}{r^i_{disc}} + \left( p - \frac{c}{r^i_{disc}} \right) \left( 1 - e^{-r^i_{disc}m} \right) \right]$$

where $b^{crf}(\tau; r^i_{disc}, \omega)$ is the value of a credit-risk-free bond with maturity $\tau$, when the investor’s rate of discount is $r^i_{disc}$ and the bond’s maturity, coupon and principal satisfy the capital structure $\omega$, as defined in equation F7.

The payoff to shareholders upon default is the residual value of the liquidated assets:

$$E_0 = E(v_0; \lambda, \omega, \theta_t) = \max \left\{ 0, \alpha V^B_t - D^{crf}(r^i_{disc}, \omega) \right\}$$

(G6)

Replacing $E_0$ with the residual value above in equation G4 evaluated at $j = 1$ yields

$$\begin{bmatrix} q_u & q_m & q_d \end{bmatrix} \begin{bmatrix} \hat{E}_3 \\ \hat{E}_2 \\ \hat{E}_1 \end{bmatrix} \max \left\{ 0, \alpha V^B_t - D^{crf}(r^i_{disc}, \omega) \right\} + \psi_1 = 0$$

(G7)

G.3 Discretized PDEs in Matrix Form

I rewrite equations G5 and G7 as follows

$$0 = \begin{bmatrix} q_m & q_d \\ \equiv q_{N-1} \end{bmatrix} \begin{bmatrix} \hat{E}_{N-1} \\ \hat{E}_{N-2} \\ \hat{E}_{N-3} \end{bmatrix} + \left\{ q_u E^{crf}(v_N; V^B_t , \omega) + \psi_{N-1} \right\}$$

and

$$0 = \begin{bmatrix} q_u & q_m \\ \equiv q_1 \end{bmatrix} \begin{bmatrix} \hat{E}_3 \\ \hat{E}_2 \\ \hat{E}_1 \end{bmatrix} + \left\{ \psi_1 + q_d \max \left( 0, \alpha V^B_t - D^{crf}(r^i_{disc}, \omega) \right) \right\}$$
This allows me to write the system of equations in \( G4 \) in matrix form

\[
\begin{bmatrix}
q_{N-1} & 0 & \cdots & 0 \\
q & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & q_1 \\
\end{bmatrix}
\begin{bmatrix}
\bar{E}^\lambda_{N-1} \\
\bar{E}^\lambda_{N-2} \\
\vdots \\
\bar{E}^\lambda_2 \\
\bar{E}^\lambda_1 \\
\end{bmatrix}
+ \begin{bmatrix}
q_u E_{crf} (v_N, V_B, \omega) + \psi_{N-1} \\
\psi_{\text{max}} (0, \alpha V_B - D_{crf} (r^i_{\text{disc}}, \omega)) \\
\end{bmatrix} = 0
\]

This gives

\[
\bar{E}^\lambda = -A^{-1} \Psi
\]

(G8)

3.1 Specializing the Method to Solve the Stochastic Volatility Model

The pre-volatility shock equity value function can be computed via equation (G8) by adjusting parameters and coefficients as follows

\[
b_{crf} (\tau, r^i_{\text{disc}}, \omega) \equiv \frac{c}{r^i_{\text{disc}}} + e^{-r^i_{\text{disc}} \cdot \tau} \left( p - \frac{c}{r^i_{\text{disc}}} \right)
\]

\[
D_{crf} (r^i_{\text{disc}}, \omega) \equiv \mu_b \left[ \frac{c \cdot m}{r^i_{\text{disc}}} + \left( p - \frac{c}{r^i_{\text{disc}}} \right) \left( 1 - e^{-r^i_{\text{disc}} \cdot m} \right) \right]
\]

\[
E_{crf} (v; V_B, \omega) \equiv V_B e^v + \mu_b \left[ \frac{(1 - \pi) (c \cdot m) + b_{crf} (m; r^i_{\text{disc}}, \omega) - p}{r} \right]
\]

\[
\begin{align*}
\tau_{\text{grow}} &= r - \delta, \\
r^i_{\text{disc}} &= r + \xi \cdot \sigma^2, \\
\nu &= r - \delta - \frac{1}{2} \sigma^2, \\
\psi_j &= \delta V_B e^{\psi_j} - \mu_b \left[ (1 - \pi) (c \cdot m) + d \left( V_B e^{\psi_j}, m; V_B, \lambda, \omega, \theta \right) - p \right] \\
&+ \lambda E_{\pi} (V_B e^{\psi_j}, 0, \omega, \theta), \quad j = 1, 2, \ldots, N - 1
\end{align*}
\]

App. H. Pinning down the Optimal Capital Structure

When the firm is free to choose the measure of outstanding bonds \( \mu_b \) as well as the coupon ratio \( c/p \), the choice of the optimal capital structure yields infinite solutions. This is so because the functions for the values of debt, equity and the optimal bankruptcy barrier are homogeneous of degree zero in \( 1/\mu_b \) and \( c \) and \( p \), as stated in the lemma below.

LEMMA 1: The optimal default barrier, debt and equity functions are homogenous of degree zero in \( c, p, \frac{1}{\mu_b} \):

\[
V^B (\lambda, \omega, \theta) = V^B (\lambda, \omega', \theta)
\]

\[
D \left( V; V^B (\lambda, \omega, \theta), \lambda, \omega, \theta \right) = D \left( V; V^B (\lambda, \omega', \theta), \lambda, \omega', \theta \right)
\]

\[
E \left( V; V^B (\lambda, \omega, \theta), \lambda, \omega, \theta \right) = E \left( V; V^B (\lambda, \omega', \theta), \lambda, \omega', \theta \right)
\]
for all $V \geq V^B(\lambda, \omega, \theta)$ and for all $\phi > 0$ such that
\[
\omega \equiv (\mu_b, m, c, p); \quad \omega' \equiv \left(\frac{1}{\phi} \cdot \mu_b, m, \phi \cdot c, \phi \cdot p\right)
\]

Proof. The homogeneity of degree zero can be readily verified in the formulas for the bond prices $d_{\sigma}(\cdot)$ (Theorem 1), equity value $E_{\sigma}(\cdot)$ and optimal default barrier $V^B_{\sigma}(\cdot)$ (Theorem 4) when volatility is constant. From the homogeneity of $d_{\sigma}(\cdot)$, it follows that the pre-volatility-shock bond price function $d(\cdot)$ (Theorem 2) also is homogenous of degree zero in $\frac{1}{\mu_b}$ $c$ and $p$. Homogeneity of $d(\cdot)$ and PDE G1 in turn imply that the pre-volatility-shock equity function satisfies the condition as well. Finally, the homogeneity of the pre-volatility-shock default barrier follows from the homogeneity of the pre-volatility-shock bond price and equity functions.

To pin down a solution, I normalize the measure of bonds $\mu_b$ to 1 and require that the principal $p$ be chosen so that debt is issued at par:

1. Without loss of generality, the measure of bonds $\mu_b$ can be normalized to 1 by lemma 1 above;

2. For a given maturity and coupon value, the bond principal is chosen so that debt is issued at par at time 0:
\[
p = \frac{1}{m \cdot \mu_b} D_{\sigma}(V_0; V^B(\omega, \theta), \omega, \theta)
\]

(H1)

where $V^B(\omega, \theta)$ is the optimal default barrier defined in Theorem 4.

The second restriction above allows me to define the optimal bond principal $p$ as an implicit function of the bond’s coupon $c$ and maturity $m$. For a given $m$, management chooses $c$ that maximizes the total firm value:
\[
c^*(m) = \arg \max_{c(m) \in R_+ \rightarrow R_+} \left\{ D_{\sigma}(V_0; V^B(m, c(m); p^*(m, c(m); \theta), m, c(m), \omega, \theta) + E_{\sigma}(V_0; V^B(m, c(m); p^*(c(m); m, \theta), m, c(m), \omega, \theta) \right\}
\]

(H2)

Restrictions 1 and 2 and equation H2 then define the optimal coupon $c$ as an implicit function of the bond maturity $m$. Lastly, management chooses $m$ that maximizes the sum of debt and equity, but this final step is omitted in the analysis for simplicity.

Appendix I. Computing the Misrepresentation Payoffs

The value of equity when a safe or risky firm misrepresents its type can be computed numerically. To do so, I solve slightly modified versions of the PDEs for the constant- and stochastic-volatility equity functions using the Finite Differences method. For ease of exposition, I suppress function arguments below and replace them with the firm type alone, $\gamma_i \equiv (\lambda_i, \sigma_{i,h})$, for $i \in \{s, r\}$, whenever there is no confusion.

Let $V^B(j)$ be type-$j$’s optimal default barrier, and denote by $c^*(j)$ and $p^*(j)$ type-$j$’s optimal coupon and principal. Similarly, let $d^*(V_i, m; \gamma_j)$ be the price of type-$j$’s newly issued bonds when the bankruptcy barrier is $V^B(j)$ and the capital structure is set optimally to $\omega^*(j) \equiv (\mu_b^*(j), m^*(j), c^*(j), p^*(j))$. The equity value of a firm of type-$i$ that mimics the capital structure of a type-$j$ firm is denoted by $\hat{E}(V_i; i \rightarrow j)$. This function can be calculated by solving a modified
version of equations 11 and 12, where the aggregate coupon and debt rollover costs are those of type- \( j \), which type- \( i \) is emulating.

\[
(r + \lambda_i) \tilde{E}(V; i \rightarrow j) = r_{\text{grow}} V_t \frac{\partial}{\partial V} \tilde{E}(V; i \rightarrow j) + \frac{1}{2} \sigma^2 V_t^2 \frac{\partial^2}{\partial V^2} \tilde{E}(V; i \rightarrow j) \\
+ \delta V_t - (1 - \pi) C^*(j) + \lambda_i \mathbb{E}^\pi(V; 0, \omega^*(j), \theta_{i,h}) \\
+ \text{type-} j \text{'s debt rollover gain/loss}
\]

(11)

where \( r_{\text{grow}} \equiv r - \delta \), \( C^*(j) = \mu^*_h(j) \cdot m^*(j) \cdot c^*(j) \) per formula 2, and \( \theta_{i,h} \equiv (r_{\text{grow}}, r_{\text{disc}}, \theta_{i,h}) \).

If the misrepresenting firm is of the safe type, \( \lambda_i = 0 \), and the PDE above is adapted from equation 11. Alternatively, in the case of a risky firm, for which \( \lambda > 0 \) and \( \sigma_h > \sigma_i \), PDE 11 is a variant of equation 12. Once the shock arrives, the volatility of the misrepresenting firm jumps to \( \sigma_h \). From then on, the value of equity is given by the constant-volatility equity formula, where (i) \( \sigma = \sigma_h \), (ii) the capital structure is that of a safe-type firm, and (iii) the post-shock default barrier is optimally set according to the formula in Theorem 4. In either case, the aggregate coupon and debt rollover cost terms are those of a firm of the opposite type.

I solve equation 11 numerically, by adapting the equity PDEs’ boundary conditions and then applying the Finite Differences method explained in Appendix G.2. Because the bankruptcy risk vanishes as the value of assets increases, the equity function still converges to a linear function of \( V \). However, while the risk-management costs and the underlying assets’ growth rate parameters are those of a type- \( i \) firm, the coupon and principal values come from type- \( j \)’s capital structure. A direct application of Claim 5 in Appendix F.2 gives

\[
\lim_{V \uparrow \infty} \tilde{E}(V; i \rightarrow j) = V + \frac{-(1 - \pi) C^*(j) + b^{crf}(m; r_{\text{disc}}, \omega^*(j)) - p^*(j)}{r}
\]

\[
= \lim_{V \uparrow \infty} E(V; \omega^*(j), \gamma_j)
\]

(12)

where \( b^{crf}(\tau; r_{\text{disc}}, \omega) \) is the value of a credit-risk-free bond with time-to-maturity \( \tau \), when the bond investors’ rate of discount is \( r_{\text{disc}} \) and the issuing firm’s capital structure is given by \( \omega \), as defined in equation F7. Finally, the lower boundary value is given by equation I3 in Appendix G.2.2, where the book value of debt is that of a type-j firm. The payoff to shareholders upon default is then

\[
\tilde{E}(V^B; i \rightarrow j) = \max \left\{ 0, \alpha V^B - D^{crf}(r_{\text{disc}}, \omega^*(j)) \right\}
\]

(13)

The bankruptcy barrier of the misrepresenting firm \( i \) must not be lower than \( V^B(j) \). Indeed, because the misrepresenting firm is indistinguishable from a type- \( j \) firm, when its assets hit \( V^B(j) \) investors consider it on default and refuse to rollover its debt. Moreover, because the dynamics of type- \( i \)’s underlying assets differ from that of type- \( j \), it may well be the case that the misrepresenting firm chooses to default sooner. Therefore, to back-out \( V^B(i \rightarrow j) \), I numerically search for \( V^B > V^B(j) \) satisfying

\[
\lim_{V \downarrow V^B} \tilde{E}^\pi(V; i \rightarrow j) = 0 \quad (\text{limited liability})
\]

\[
\lim_{V \downarrow V^B} \frac{\partial}{\partial V} \tilde{E}(V; i \rightarrow j) = 0 \quad (\text{smooth-pasting})
\]

If there is no such value, \( V^B(i \rightarrow j) \) is set to \( V^B(j) \).
Appendix J. Pricing in a Pooling Market Outcome

This section discusses the approach used in the computation of the payoffs in a pooling market outcome. I first present the proof to Proposition 2. Next, I derive the unobservable-type firms’ equity function PDE, and discuss the numerical method used to back out the equity functions and optimal default barriers.

J.1 Time-varying measure of safe firms

Proposition 2 states that the ratio of safe-to-risky firms among undisclosed-type firms in the same cohort depends on the age of the cohort and the initial measure of safe firms alone.

Proof. Let $\mu^\text{pos}_i(a)$ denote the measure of pre-volatility-shock firms of type-$i$ in a cohort of age $a$. At the start of a cohort, all firms are pre-volatility-shock firms. Therefore,

$$\mu^\text{pos}_s(0) = \mu_s, \quad \mu^\text{pos}_r(0) = 1 - \mu_s;$$

Let $t^\text{vb}_i$ denote the upper barrier first-passage time for a pre-volatility-shock firm of type $i$:

$$t^\text{vb}_i \equiv \inf \{ u > 0 : V^*_i(u) \leq V^B(u|\mu_b, \gamma, \mu_s) \}$$

The probability that this first-passage time process time occurs in the interval $(a, a')$ given that it has not yet occurred at time $a$ is:

$$p^\text{vb}_i(a, a') \equiv \Pr (t^\text{vb}_i \leq a' | t^\text{vb}_i > a)$$  \hspace{1cm} (J1)

Because the underlying value of assets of a pre-volatility-shock firm follows the same geometric Brownian motion process, the safe- and risky-type’s first-passage times are identically distributed. It follows that the type-contingent probabilities $p^\text{vb}_i(\cdot, \cdot)$ coincide:

$$p^\text{vb}_s(a, a') = p^\text{vb}_r(a, a') \quad \forall a' > a \geq 0$$

Disregarding the occurrence of volatility shocks for a moment, the measure of pre-volatility-shock firms of type-$i$ evolves as

$$\mu^\text{pos}_i(a + da) = \left[ 1 - p^\text{vb}_i(a, a + da) \right] \cdot \mu^\text{pos}_i(a)$$

in an interval of measure $da$. Thus, by equation J1 above, the ratio of safe to risky firms is unaffected by the process of firm type revelation that happens in the crossing of the upper barrier:

$$\frac{\mu^\text{pos}_s(a + da)}{\mu^\text{pos}_s(a)} = \frac{\mu^\text{pos}_r(a + da)}{\mu^\text{pos}_r(a)} \quad \forall a \geq 0$$

In other words, while the crossing of the upper barrier reveals private information about a firm, it does not affect the firm-type distribution among the pre-volatility-shock firms of the same cohort.

Now let us consider the occurrence of volatility shocks. The probability that a pre-volatility-shock risky firm suffers a volatility shock in an infinitesimal interval $da$ is simply $\lambda \cdot da$. Let $t^\sigma$ be the first arrival time of a volatility shock. Since the shock process is independent of value of the process governing the value of the firm’s underlying assets, $V$, so are the stopping times $t^\text{vb}_r$ and $t^\sigma$. 

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independent from one another. Therefore, the probability that the type of a risky, pre-volatility-shock firm gets revealed in the interval \((a, a + da)\) is simply

\[
Pr(\{t^{vb}_r \in (a, a + da]\} \cup \{t^s \in (a, a + da]\} \mid \min\{t^{vb}_r, t^s\} > a = p^{vb}_r (a, a + da) + \left[1 - p^{vb}_r (a, a + da]\right] \lambda \cdot da
\]

From this equation, it follows that the measure of risky firms in the set of pre-volatility-shock firms in a given cohort evolves according to

\[
\mu^{p vs}_r (a + da) = \mu^{p vs}_r (a) × \left\{1 - \left[p^{vb}_r (a, a + da) + \left[1 - p^{vb}_r (a, a + da]\right] \lambda \cdot da\right]\right\}
\]

\[
= \mu^{p vs}_r (a) \cdot \left[1 - p^{vb}_r (a, a + da]\right] \cdot (1 - \lambda \cdot da)
\]

(J3)

Denoting the ratio of safe to risky pre-volatility-shock firms by \(r^s_r (a)\), we have

\[
r^s_r (a + da) = \frac{\mu^{p vs}_s (a + da)}{\mu^{p vs}_r (a + da)}
\]

\[
= \frac{\mu^{p vs}_s (a) \cdot \left[1 - p^{vb}_s (a, a + da]\right]}{\mu^{p vs}_r (a) \cdot \left[1 - p^{vb}_r (a, a + da]\right] \cdot (1 - \lambda \cdot da)}
\]

\[
= \frac{r^s_r (a) \cdot (1 - \lambda \cdot da)}{1 - \lambda \cdot da}
\]

(J4)

since \(p^{vb}_s (a, a + da)\) and \(p^{vb}_r (a, a + da)\) coincide. Thus, \(\frac{d}{da} r^s_r (a) = \lambda r^s_r (a)\), which gives

\[
r^s_r (a) = r^s_r (0) e^{\lambda a}
\]

(J5)

The results follows from equation J5 and \(r^s_r (0) = \mu_s \cdot (1 - \mu_s)^{-1}\):

\[
p_s (a) \equiv \frac{\mu^{p vs}_s (a)}{\mu^{p vs}_r (a) + \mu^{p vs}_r (a)} = \frac{\mu_s}{\mu_s + (1 - \mu_s) \cdot e^{-\lambda a}}
\]

(J6)

Q.E.D.

\[\square\]

J.2 Equity Price in a Pooling Market Outcome

As discussed in section V.D, I set the share of safe firms among the undisclosed-type firms to \(\mu_s\), regardless of the age of the cohort: \(p_s (a) = \mu_s\) for all \(a \geq 0\). This assumption renders the bankruptcy barriers constant and greatly facilitates the derivation of equity prices in a pooling market outcome. Let then \(V^{B}_{p,j}\) denote bankruptcy barrier of an undisclosed-type firm of type-\(j\), for \(j \in \{s, r\}\). The price of a bond issued by an undisclosed-type firm is the average of the type-contingent, fundamental bond prices (the prices that would hold under full information if the default barriers were set to \(V^{B}_{p,s}\) and \(V^{B}_{p,r}\)). Under the assumption above, equation 20 becomes:

\[
d \left(V_t, \tau; \mu_b, \mathcal{b}^{EP}, \gamma, \mu_s\right) = \mu_s \times d \left(V_t, \tau; V^{B}_{p,s}, \mu_b, \mathcal{b}^{EP}, \gamma_s\right) + (1 - \mu_s) \times d \left(V_t, \tau; V^{B}_{p,r}, \mu_b, \mathcal{b}^{EP}, \gamma_r\right)
\]

(J7)

for a common measure of outstanding bonds, \(\mu_b\), and bond contract, \(\mathcal{b}^{EP} \equiv (\mathcal{m}^{EP}, \mathcal{e}^{EP}, \mathcal{p}^{EP})\).
Similarly to the full information case, the PDE of the equity price function of an undisclosed-type is given by

$$(r + \lambda_i) E \left( V_t; \mu_b, b^{EP}, \gamma, \mu_s | \gamma = \gamma_i \right) = r_{grow} V_t \frac{\partial}{\partial V} E \left( V_t; \mu_b, b^{EP}, \gamma, \mu_s | \gamma = \gamma_i \right) + 1/2 \sigma^2 V_t^2 \frac{\partial^2}{\partial V^2} E \left( V_t; \mu_b, b^{EP}, \gamma, \mu_s | \gamma = \gamma_i \right) + \delta V_t - (1 - \pi) (\mu_b \cdot m \cdot c^{EP}) + \lambda_i E^{\sigma} \left( V_t; \mu_b, b^{EP}, \theta_{i,h} \right) + d \left( V_t, \tau; \mu_b, b^{EP}, \gamma, \mu_s \right) - p^{\sigma f}$$

for $i \in \{s, r\}$, with $\lambda_r > \lambda_s = 0$ and $\sigma_{r,h} > \sigma_{s,h} = \sigma_l$. At the moment a risky, undisclosed-type firm is hit by a shock, creditors become aware of its new risk-exposure and adjust the valuation of its bonds accordingly. This is captured by the term $\lambda_i E^{\sigma} (\cdot; \mu_b, b^{EP}, \theta_{i,h})$ in the expression above, where $E^{\sigma} (\cdot)$ denotes the constant-volatility equity function when the bankruptcy barrier is set optimally (derived in Appendix F.)

Turning to the derivation of the boundary conditions, a direct application of Claims 5 and 9 gives

$$\lim_{V \uparrow \infty} E \left( V_t; \mu_b, b^{EP}, \gamma, \mu_s | \gamma = \gamma_i \right) = V + \mu_b - (1 - \pi) \cdot c^{EP} \cdot m^{EP} + b^{\sigma f} \left( m^{EP}; r_{disc}, \mu_b, b^{EP} \right)$$

where $b^{\sigma f} (m^{EP}; r_{disc}, \mu_b, b^{EP})$ denotes the price of a newly-issued, standardized, credit-risk-free bond. Therefore, the upper boundary values for safe and risky firms coincide. The computation of the lower boundary conditions is a bit more involved, however. Optimality of the bankruptcy decision for the type that defaults sooner (type with the highest $V^B$ value) presupposes the usual smooth-pasting and limited liability conditions. If $V^B_{p,j} > V^B_{p,i}$, type-$j$'s equity function must satisfy

$$\lim_{V \downarrow V^B_{p,j}} \frac{\partial}{\partial V} E \left( V_t; \mu_b, b^{EP}, \gamma, \mu_s | \gamma = \gamma_j \right) = 0 \quad \text{(smooth-pasting)}$$

$$\lim_{V \downarrow V^B_{p,j}} E \left( V_t; \mu_b, b^{EP}, \gamma, \mu_s | \gamma = \gamma_j \right) = 0 \quad \text{(limited liability)}$$

In the case of the type-$i$, however, the equity function is discontinuous at $V^B_{p,j}$ because the firm remains alive. Once the underlying value of assets $V$ of a type-$i$ firm crosses type-$j$'s bankruptcy barrier from above for the first time, the asymmetry of information vanishes. From then on, the bonds are priced by the appropriate full-information bond formula derived in Theorem 1 or Theorem 2, depending on whether the firm is subject to volatility shocks. Therefore, the lower boundary condition for type-$i$ firms is

$$\lim_{V \downarrow V^B_{p,j}} \frac{\partial}{\partial V} E \left( V_t; \mu_b, b^{EP}, \gamma, \mu_s | \gamma = \gamma_j \right) = \begin{cases} E^{\sigma} \left( V^B_{p,j}; \mu_b, b^{EP}, \theta \right), & \text{if } i = s \\ E \left( V^B_j; V^B_i, \lambda_r, \omega, \theta_h \right), & \text{otherwise} \end{cases}$$

where $E^{\sigma} (\cdot; \mu_b, b^{EP}, \theta)$ is the safe-type's constant-volatility equity function under full-information, and $E \left( V^B_j; V^B_i, \lambda_r, \omega, \theta_h \right)$ is the solution to equation G1 when the pre-volatility shock default barrier is set optimally to $V^B_i \equiv V^B (\mu_b, b^{EP}, \gamma_r)$. Either way, if type-$i$ defaults last, its default barrier must coincide with the corresponding full-information, optimal bankruptcy policy discussed in the previous sections.
Finally, the equity function PDEs can be solved using the numerical approach outlined in Appendix G, for arbitrary $V_{p,s}^B$ and $V_{p,r}^B$ values. To back out the optimal bankruptcy barriers, I proceed as follows. Assuming type-$j$ defaults first, I set $V_{p,i}^B$ to the type-$i$’s full information, optimal bankruptcy barrier, and search for $V_{p,j}^B > V_{p,i}^B$ that satisfies the smooth-pasting and limited-liability conditions. I do so for both types, but the search for the higher bankruptcy barrier yields a result only when the risky type defaults first (Typically, there is no value $V_{p,s}^B > V_{p,r}^B$ that satisfies the lower boundary conditions.) The algorithm used in the search for the optimal barriers is outline in the Online Appendix D.